Graph theory terminology

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A **graph** is a pair $G = (V, E)$ where V is the set of **vertices** and E is the set of **edges**. An **edge** is an unordered pair of vertices. Two vertices joined by an edge are said to be **adjacent**. Two vertices are **neighbors** if they are adjacent. The **degree** $\deg(v)$ of vertex v is the number of its neighbors. A graph is **regular** of degree r if all vertices have degree r.

Exercise 1. Prove: $\sum_{v \in V} \deg(v) = 2|E|$.

The number of vertices will usually be denoted by n.

Exercise 2. Observe: $|E| \le \binom{n}{2}$.

Complete graphs, complete bipartite graphs, subgraphs

In a **complete graph**, all pairs of vertices are adjacent. The complete graph on n vertices is denoted by K_n. It has $\binom{n}{2}$ edges.

The vertices of a **complete bipartite graph** are split into two subsets $V = V_1 \dot\cup V_2$; and $E = \{\{x, y\} : x \in V_1, y \in V_2\}$. If $k = |V_1|$ and $\ell = |V_2|$ then we obtain the graph $K_{k,\ell}$. This graph has $n = k + \ell$ vertices and $|E| = k\ell$ edges.

The graph $H = (W, F)$ is a **subgraph** of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$.

Exercise+ 3. (Mandel–Turán) Prove: if G is triangle-free ($K_3 \not\subseteq G$) then $|E| \le \lfloor n^2/4 \rfloor$. Show that this bound is tight for every n.

Walks, paths, cycles, trees

- **walk** of length k: a sequence of $k + 1$ vertices $v_0, \dots, v_k$ such that v_{i-1} and v_i are adjacent for all i.
- **path:** a walk without repeated vertices. P_{k+1} denotes a path of length k (it has $k + 1$ vertices)
- **closed walk** of length k: a walk $v_0, \dots, v_k$ where $v_k = v_0$.
- **cycle of length** k or k-**cycle**: a closed walk of length k with no repeated vertices except that $v_0 = v_k$. Notation: C_k.
- a graph G is **connected** if there is a path between each pair of vertices.
- a **tree** is a connected graph without cycles.

Exercise+ 4. (Kővári–Sós–Turán) Prove: if G has no 4-cycles ($C_4 \not\subseteq G$) then then $|E| = O(n^{3/2})$. Show that this bound is tight.

Exercise 5. Prove that every tree has $n - 1$ edges.

Cliques, distance, diameter, chromatic number

- A k-**clique** is a subgraph isomorphic to K_k (a set of k pairwise adjacent vertices).
- An **independent set** of size k is the complement of a k-clique: k vertices, no two of which are adjacent.
- The **distance** $\text{dist}(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path between them. If there is no path between x and y then their distance is said to be infinite: $\text{dist}(x, y) = \infty$.
- The **diameter** of a simple graph is the maximum distance between all pairs of vertices. So if a graph has diameter d then $(\forall x, y \in V)(\text{dist}(x, y) \leq d)$ and $(\exists x, y \in V)(\text{dist}(x, y) = d)$.
- The **girth** of a graph is the length of its shortest cycle. If a graph has no cycles then its girth is said to be infinite.

  Examples (verify!): trees have infinite girth; the $m \times n$ grid (see figure) has girth 4 if $m, n \geq 2$; $K_{m,n}$ has girth 4 if $m, n \geq 2$, K_n has girth 3.
- A **legal** k-**coloring** of a graph is a function $c : V \to [k] = \{1, \ldots, k\}$ such that adjacent vertices receive different colors, i. e., $\{u, v\} \in E \Rightarrow c(u) \neq c(v)$. A graph is k-**colorable** if there exists a legal k-coloring. The **chromatic number** $\chi(G)$ of a graph is the smallest k such that G is k-colorable.
- A graph is **bipartite** if it is 2-colorable.

Exercise 6. Prove: the bipartite graphs are exactly the subgraphs of the complete bipartite graphs.

Exercise 7. Prove: a graph is bipartite if and only if it has no odd cycles.

Exercise 8. Prove: if every vertex has degree $\leq d$ then $\chi(G) \leq d + 1$.

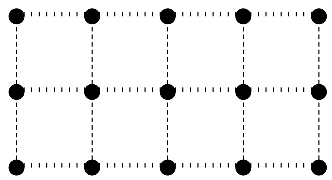
Exercise* 9. Prove: $(\forall k)(\exists G)(\chi(G) \geq k$ and G is triangle-free.)

Exercise 10. (Erdős)** Prove: $(\forall k, g)(\exists G)(\chi(G) \geq k$ and G has girth $\geq g$.)

Exercise 10. Prove: if G is regular of degree r and G has diameter 2 then $n \leq r^2 + 1$.

Exercise 11. Prove: if G is regular of degree r and G has girth ≥ 5 then $n \geq r^2 + 1$. Show that $n = r^2 + 1$ is possible for $r = 1, 2, 3$.

Exercise* 12. (Hoffmann–Singleton) Prove: if G is regular of degree r and G has girth ≥ 5 and $n = r^2 + 1$ then $r \in \{1, 2, 3, 7, 57\}$.



Picture: the 3×5 grid.