1. Review the fact that the volume of the parallelopiped spanned by \( n \) vectors in \( \mathbb{R}^n \) is the absolute value of the determinant formed by those vectors.

2. Let \( s_n \) denote the \((n\text{-dimensional})\) volume of a simplex determined by \( n \) linearly independent vectors in \( \mathbb{R}^n \). Let \( p_n \) be the \((n\text{-dimensional})\) volume of the parallelopiped generated by the same \( n \) vectors. Show that

\[
\frac{s_n}{p_n} = \frac{1}{n!}
\]

3. Challenge Problem: \( n \)-dimensional Pythagorean Theorem for hyperplanes

For \( 1 \leq i \leq n \), let \( \pi : \mathbb{R}^n \to \mathbb{R}^n \) denote the projection to the \( i \)-th coordinate hyperplane (this projection simply sets the \( i \)-th coordinate of every point to zero).

Let \( A \) be a hyperplane in \( \mathbb{R}^n \) and \( F \subset A \) be a measurable set (relative to the Lebesgue measure on \( A \)). Let \( F_i = \pi_i(F) \) be the \( i \)-th projection of \( F \). Let \( V \) denote the \((n - 1)\)-dimensional volume of \( F \) (relative to the subspace \( A \)); and let \( V_i \) denote the \( n - 1 \)-dimensional volume of \( F_i \). Prove:

\[
V^2 = \sum_{i=1}^{n} V_i^2.
\]

4. Challenge Problem: \( n \)-dimensional Pythagorean Theorem for \( k \)-dimensional subspaces

For \( T \subseteq [n] \), let \( \pi_T : \mathbb{R}^n \to \mathbb{R}^n \) denote the projection to the coordinate subspace spanned by the standard basis vectors \( \{e_i : i \in T\} \), i.e., \( \pi_T \) sets all coordinates outside \( T \) to zero.

Let \( A \) be a \( k \)-dimensional subspace of \( \mathbb{R}^n \). Let \( F \subset A \) be a measurable set (relative to the Lebesgue measure on \( A \)). Let \( V \) denote the \( k \)-dimensional volume of \( F \). For \( T \subseteq [n], |T| = k \), let \( V_T \) denote the \( k \)-dimensional measure of \( \pi_T(F) \). Prove:

\[
V^2 = \sum_T V_T^2,
\]

where the summation extends over the \( \binom{n}{k} \) \( k \)-subsets \( T \subseteq [n] \).
5. Prove the following:

(a) If \( a_n \sim b_n \) and \( a_n > 1.001 \) then \( \ln(a_n) \sim \ln(b_n) \).
(b) If \( a_n = \Theta(b_n) \) and \( a_n \to \infty \) then \( \ln(a_n) \sim \ln(b_n) \).

6. Let \( p_n \) denote the \( n \)th prime number. Show that the relation \( p_n \sim n \ln(n) \) implies the Prime Number Theorem.

7. In this exercise, do not use Stirling’s formula. For \( 1 \leq k \leq n \), prove:

(a) \( \binom{n}{k}^k \leq \binom{n}{k} < \left( \frac{en}{k} \right)^k \).
(b) \( \binom{n}{k} + \binom{n}{k-1} + \ldots + \binom{n}{0} < \left( \frac{en}{k} \right)^k \).

8. Let \( 0 < \alpha < 1 \). The entropy function \( H(\alpha) \) is defined as

\[
H(\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2 (1 - \alpha).
\]

Prove:

(a) \( 0 < H(\alpha) \leq 1 \);
(b) \( H(\alpha) = 1 \iff \alpha = 1/2 \);
(c) \( \lim_{\alpha \to 0} H(\alpha) = \lim_{\alpha \to 1} H(\alpha) = 0 \).
(d) \( H(\alpha) = H(1 - \alpha) \).

9. **Asymptotics of binomial coefficients** Prove: if \( \{k_n\} \) is a sequence of positive integers such that \( \lim_{n \to \infty} k_n / n = \alpha \) where \( 0 < \alpha < 1 \) then

\[
\log_2 \left( \frac{n}{k_n} \right) \sim H(\alpha).
\]

In other words,

\[
\binom{n}{k_n} = 2^{H(\alpha)n(1+o(1))}.
\]

*Hint:* Stirling’s formula.

10. **Chromatic number and independence number.** Let \( G \) be a graph on \( n \) vertices, and let \( \chi(G) \) and \( \alpha(G) \) denote the chromatic number of \( G \) and the size of a maximal independent set of \( G \), respectively. (An independent set is a subset of the vertex set which includes no edges.) Show that \( \chi(G) \cdot \alpha(G) \geq n \).

11. * (Eventown) Given a set \( X \) with \( n \) elements, and a family \( \mathcal{F} \) of subsets of \( X \) such that, given any two elements \( A, B \) (not necessarily different) of \( \mathcal{F} \), \( A \cap B \) contains an even number of elements, show that there are at most \( 2^{\lceil \frac{n}{2} \rceil} \) elements in \( \mathcal{F} \).