Discrete Mathematics REU
Problems – June 20–22, 2001

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1. Review the fact that the volume of the parallelopiped spanned by n vectors in $\mathbb{R}^n$ is the absolute value of the determinant formed by those vectors.

2. Let s_n denote the (n-dimensional) volume of a simplex determined by n linearly independent vectors in $\mathbb{R}^n$. Let p_n be the (n-dimensional) volume of the parallelopiped generated by the same n vectors. Show that

$$\frac{s_n}{p_n} = \frac{1}{n!}$$

3. **Challenge Problem: n-dimensional Pythagorean Theorem for hyperplanes** For $1 \le i \le n$, let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ denote the projection to the i-th coordinate hyperplane (this projection simply sets the i-th coordinate of every point to zero).

   Let A be a hyperplane in $\mathbb{R}^n$ and $F \subset A$ be a measurable set (relative to the Lebesgue measure on A). Let $F_i = \pi_i(F)$ be the i-th projection of F. Let V denote the $(n-1)$-dimensional volume of F (relative to the subspace A); and let V_i denote the $n-1$-dimensional volume of F_i. Prove:

$$V^2 = \sum_{i=1}^{n} V_i^2.$$

4. **Challenge Problem: n-dimensional Pythagorean Theorem for k-dimensional subspaces**

   For $T \subseteq [n]$, let $\pi_T : \mathbb{R}^n \to \mathbb{R}^n$ denote the projection to the coordinate subspace spanned by the standard basis vectors $\{\mathbf{e_i} : i \in T\}$, i.e., π_T sets all coordinates outside T to zero.

   Let A be a k-dimensional subspace of $\mathbb{R}^n$. Let $F \subset A$ be a measurable set (relative to the Lebesgue measure on A). Let V denote the k-dimensional volume of F. For $T \subseteq [n]$, $|T| = k$, let V_T denote the k-dimensional measure of $\pi_T(F)$. Prove:

$$V^2 = \sum_{T} V_T^2,$$

   where the summation extends over the $\binom{n}{k}$ k-subsets $T \subseteq [n]$.

5. Prove the following:

   (a) If $a_n \sim b_n$ and $a_n > 1.001$ then $\ln(a_n) \sim \ln(b_n)$.

   (b) If $a_n = \Theta(b_n)$ and $a_n \to \infty$ then $\ln(a_n) \sim \ln(b_n)$.

6. Let p_n denote the nth prime number. Show that the relation $p_n \sim n\ln(n)$ implies the Prime Number Theorem.

7. In this exercise, do not use Stirling's formula. For $1 \le k \le n$, prove:

   (a)
$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} < \left(\frac{\mathrm{e}n}{k}\right)^k.$$

   (b)
$$\binom{n}{k} + \binom{n}{k-1} + \ldots + \binom{n}{0} < \left(\frac{\mathrm{e}n}{k}\right)^k.$$

8. Let $0 < \alpha < 1$. The **entropy function** $H(\alpha)$ is defined as
$$H(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha).$$
   Prove:

   (a) $0 < H(\alpha) \le 1$;

   (b) $H(\alpha) = 1 \iff \alpha = 1/2$;

   (c) $\lim_{\alpha\to 0} H(\alpha) = \lim_{\alpha \to 1} H(\alpha) = 0$.

   (d) $H(\alpha) = H(1-\alpha)$.

9. **Asymptotics of binomial coefficients** Prove: if $\{k_n\}$ is a sequence of positive integers such that $\lim_{n\to\infty} k_n/n = \alpha$ where $0 < \alpha < 1$ then
$$\log_2 \binom{n}{k_n} \sim H(\alpha).$$
   In other words,
$$\binom{n}{k_n} = 2^{H(\alpha)n(1+o(1))}.$$
   Hint: Stirling's formula.

10. **Chromatic number and independence number.** Let G be a graph on n vertices, and let $\chi(G)$ and $\alpha(G)$ denote the chromatic number of G and the size of a maximal independent set of G, respectively. (An independent set is a subset of the vertex set which includes no edges.) Show that $\chi(G) \cdot \alpha(G) \ge n$.

11. * **(Eventown)** Given a set X with n elements, and a family $\mathcal{F}$ of subsets of X such that, given any two elements A, B (not necessarily different) of $\mathcal{F}$, $A \cap B$ contains an even number of elements, show that there are at most $2^{\lfloor \frac{n}{2} \rfloor}$ elements in $\mathcal{F}$.