DISCRETE MATHEMATICS PROBLEMS. JULY 8, 2002

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For updates, please consult the web site
`http://people.cs.uchicago.edu/~laci/reu02.dir`

Important Change: This week, lectures will be held **Monday, Tuesday, Thursday, Friday**, and the problem session will be on **Wednesday**. Each of these will be from 5:00-6:00PM.

Exercise 1. Prove: the dimension of the space of homogeneous polynomials of degree s in n variables is $\binom{n+s-1}{s}$. (A multivariate polynomial is *homogeneous* if it is a linear combination of monomials of the same degree. Example: $x^4 + 5x^3y + 13x^2yz + xz^3$ is a homogeneous polynomial of degree 4 in 3 variables. The polynomial $x^4 + xyz$ is not homogeneous.)
Hint: Count the monomials of degree exactly s in n variables.

Exercise 2. Prove: if $\mu_1, \dots, \mu_m \in \mathbb{F}^n$, $f_1, \dots, f_m \in \mathbb{F}[x_1, \dots, x_n]$, and $f_i(\mu(j)) \neq 0$ if $i = j$ and $f_i(\mu(j)) = 0$ if $i > j$, then $f_1, \dots, f_m$ are linearly independent. (Note: we have no condition for the case $i < j$.)

Definition 3. Let X be a set of size n (the "universe"), and let $\mathcal{F} = \{A_1, \dots, A_m\}$, $A_i \subseteq X$, be a family of subsets of X. We say $\mathcal{F}$ is *uniform* if, for all $1 \leq i, j \leq m$, $|A_i| = |A_j|$. Let $L \subset \mathbb{Z}$ be a finite set of non-negative integers. We say $\mathcal{F}$ is L-intersecting if, for all $1 \leq i < j \leq m$, $|A_i \cap A_j| \in L$.

Theorem 4. (Frankl–Wilson) Suppose $\mathcal{F}$ is L-intersecting. Then

$$|\mathcal{F}| \leq \binom{n}{\leq |L|} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{|L|}.$$

Theorem 5. (Ray-Chaudhuri–Wilson) Suppose $\mathcal{F}$ is uniform and L-intersecting. Then

$$|\mathcal{F}| \leq \binom{n}{|L|}.$$

In class, we proved a weaker theorem (implied by both of the above theorems): if $\mathcal{F}$ is uniform and L-intersecting, then $|\mathcal{F}| \leq \binom{n}{\leq |L|}$.

Exercise 6. Prove the Frankl-Wilson Theorem by adapting the proof given in class to the case of non-uniform set systems. *Hint:* Use Exercise 2. The polynomials used in class will have to be modified slightly, and put in the correct order.

Exercise 7. Prove the Ray-Chaudhuri–Wilson Theorem.
Hint: Let $k := |A_1| = \cdots = |A_m|$. Let $g(x_1, \dots, x_n)$ be the linear polynomial $k - \sum_{i=1}^{n} x_i$. For each subset $I \subseteq n$, define a polynomial h_I by (OVER)

$$h_I(x_1, \dots, x_n) = g(x_1, \dots, x_n) \prod_{i \in I} x_i.$$

Let $\overline{h_I}$ be the multilinearization of h_I, and let $\overline{f_1}, \dots, \overline{f_m}$ be the multilinear polynomials from the proof given in class. Use Exercise 2 to prove that the following set of polynomials is linearly independent: $\overline{f_1}, \dots, \overline{f_m}$, together with $\overline{h_I}$, for all $I \subset n$ such that $|I| \leq s-1$. Deduce that $m + \binom{n}{\leq s-1} \leq \binom{n}{\leq s}$, and so $m \leq \binom{n}{s}$.