Circulant Matrices

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Definition 1. For any given $a_0, a_1, \cdots, a_{n-1} \in \mathbb{C}$, the **circulant matrix** $B = (b_{i,j})_{n\times n}$ is defined by $b_{i,j} := a_{j-i (\text{mod } n)}$.

$$B = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix}$$

The exercise below leads to a nice and surprising formula for the determinant of a circulant matrix.

Exercise 2. Let ω be a primitive n^{th} root of unity (ω has order n). Show that all eigenvalues of the circulant matrix (B) of $a_0, \cdots, a_{n-1}$ are of the form $\lambda_\omega = \sum_{i=0}^{n-1} a_i \omega^i$, where ω is an n^{th} root of unity. Show that there is a corresponding orthonormal eigenbasis. Find the eigenvectors. (*Hint:* characters.)

Note that $\det(B) = \prod_\omega \lambda_\omega$. Next we generalize the cyclic shifts that produce a circulant to the action of any finite abelian group.

Definition 3. Given a sequence of numbers $(a_g)_{g\in G}$ where G is a finite abelian group of order n, the **G-circulant matrix** $B = (b_{g,h})_{n\times n}$ is defined by $b_{g,h} = a_{gh^{-1}}$, $g, h \in G$ (the rows and columns of B are indexed by elements of G).

Example 4. The circulant matrix (Definition 1) is a G-circulant with $G = \mathbb{Z}_n$.

Example 5. If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, the circulant matrix has the form

$$B = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$$

(Verify!)

Exercise 6. Prove that all eigenvalues of the G-circulant matrix with first row $(a_g)_{g\in G}$ are of the form $\lambda_\chi = \sum_{g\in G} \chi(g) a_g$ where $\chi \in \widehat{G}$ is a character of G. Show that there is a corresponding orthonormal eigenbasis. Find the eigenvectors. (*Hint:* characters.)

Answers to Exercise 2 and Exercise 6 : $\vec{v}_\omega = (1, \omega, \omega^2, \cdots, \omega^{n-1})$ and $\vec{v}_\chi = (\chi(g))_{g \in G}$.