Graph theory terminology – Undirected graphs

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In this course, the term "graph" (without adjective) will always refer to **undirected graphs** (adjacency is a symmetric relation). (We use "digraph" when referring to directed graphs.) In this handout we discuss (undirected) graphs only.

A **graph** is a pair $G = (V, E)$ where V is the set of **vertices** and E is the set of **edges**. An **edge** is an unordered pair of vertices. Two vertices joined by an edge are said to be **adjacent**. Two vertices are **neighbors** if they are adjacent. The **degree** $\deg(v)$ of vertex v is the number of its neighbors. A graph is **regular** of degree r if all vertices have degree r.

Remark on terminology. In our terminology, a vertex is never adjacent to itself (no "loops" permitted) and there is at most one edge between any pair of vertices (no "parallel edges" permitted). An object where loops and parallel edges are permitted is called a "multigraph." We shall not discuss multigraphs in this note. Many authors use the term "graph" to indicate "multigraphs;" our "graphs" are then called "simple graphs."

Exercise 1. Prove: $\sum_{v \in V} \deg(v) = 2|E|$.

The number of vertices will usually be denoted by n.

Exercise 2. Observe: $|E| \le \binom{n}{2}$.

Complete graphs, complete bipartite graphs, subgraphs

In a **complete graph**, all pairs of vertices are adjacent. The complete graph on n vertices is denoted by K_n. It has $\binom{n}{2}$ edges.

The vertices of a **complete bipartite graph** are split into two subsets $V = V_1 \dot\cup V_2$; and $E = \{\{x, y\} : x \in V_1, y \in V_2\}$. If $k = |V_1|$ and $\ell = |V_2|$ then we obtain the graph $K_{k,\ell}$. This graph has $n = k + \ell$ vertices and $|E| = k\ell$ edges.

The graph $H = (W, F)$ is a **subgraph** of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$.

Exercise+ 3. (Mandel–Turán) Prove: if G is triangle-free ($K_3 \not\subseteq G$) then $|E| \le \lfloor n^2/4 \rfloor$. Show that this bound is tight for every n.

Walks, paths, cycles, trees

- **walk** of length k: a sequence of $k + 1$ vertices $v_0, \dots, v_k$ such that v_{i-1} and v_i are adjacent for all i.
- **path:** a walk without repeated vertices. P_{k+1} denotes a path of length k (it has $k + 1$ vertices)
- **closed walk** of length k: a walk $v_0, \dots, v_k$ where $v_k = v_0$.

- **cycle of length** k or k-**cycle**: a closed walk of length k with no repeated vertices except that $v_0 = v_k$. Notation: C_k.
- a graph G is **connected** if there is a path between each pair of vertices.
- a **tree** is a connected graph without cycles.

Exercise+ 4. **(Kővári–Sós–Turán)** Prove: if G has no 4-cycles ($C_4 \not\subseteq G$) then then $|E| = O(n^{3/2})$. Show that this bound is tight.

Exercise 5. Prove that every tree has $n - 1$ edges.

Cliques, distance, diameter, chromatic number

- A k-**clique** is a subgraph isomorphic to K_k (a set of k pairwise adjacent vertices).
- An **independent set** of size k is the complement of a k-clique: k vertices, no two of which are adjacent.
- The **distance** $\mathrm{d}ist(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path between them. If there is no path between x and y then their distance is said to be infinite: $\mathrm{d}ist(x, y) = \infty$.
- The **diameter** of a simple graph is the maximum distance between all pairs of vertices. So if a graph has diameter d then $(\forall x, y \in V)(\mathrm{d}ist(x, y) \le d)$ and $(\exists x, y \in V)(\mathrm{d}ist(x, y) = d)$.
- The **girth** of a graph is the length of its shortest cycle. If a graph has no cycles then its girth is said to be infinite.

  Examples (verify!): trees have infinite girth; the $m \times n$ grid (see figure) has girth 4 if $m, n \ge 2$; $K_{m,n}$ has girth 4 if $m, n \ge 2$, K_n has girth 3.
- A **legal** k-**coloring** of a graph is a function $c : V \to [k] = \{1, \ldots, k\}$ such that adjacent vertices receive different colors, i. e., $\{u, v\} \in E \Rightarrow c(u) \ne c(v)$. A graph is k-**colorable** if there exists a legal k-coloring. The **chromatic number** $\chi(G)$ of a graph is the smallest k such that G is k-colorable.
- A graph is **bipartite** if it is 2-colorable.

Exercise 6. Prove: the bipartite graphs are exactly the subgraphs of the complete bipartite graphs.

Exercise 7. Prove: a graph is bipartite if and only if it has no odd cycles.

Exercise 8. Prove: if every vertex has degree $\le d$ then $\chi(G) \le d + 1$.

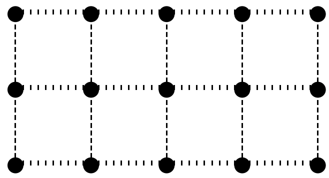
Exercise* **9.** Prove: $(\forall k)(\exists G)(\chi(G) \ge k$ and G is triangle-free.)

Exercise** **10.** **(Erdős)** Prove: $(\forall k, g)(\exists G)(\chi(G) \geq k$ and G has girth $\geq g.)$

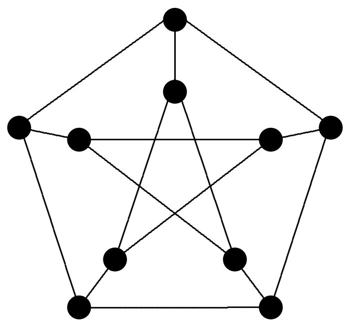
Exercise 10. Prove: if G is regular of degree r and G has diameter 2 then $n \leq r^2 + 1$.

Exercise 11. Prove: if G is regular of degree r and G has girth ≥ 5 then $n \geq r^2 + 1$. Show that $n = r^2 + 1$ is possible for $r = 1, 2, 3$.

Exercise* **12.** **(Hoffmann–Singleton)** Prove: if G is regular of degree r and G has girth ≥ 5 and $n = r^2 + 1$ then $r \in \{1, 2, 3, 7, 57\}$.



Picture: the 3×5 grid.



Picture: the Petersen graph