

DISCRETE MATHEMATICS PROBLEMS. JUNE 24, 2002

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1. RAMSEY THEORY

Definition 1. We say that m ‘arrows’ the ℓ -tuple (n, n, \dots, n) if $\forall \ell$ -colorings of the **edges** of the complete graph on m vertices, K_m , \exists a homogeneous n -clique (a K_n subgraph with all edges one color). This is written as

$$m \rightarrow \underbrace{(n, n, \dots, n)}_{\ell}.$$

Exercise 2. Prove $6 \rightarrow (3, 3)$.

Exercise 3. Prove $4^n \rightarrow (n+1, n+1)$.

Hint: Choose a pivot vertex v_0 . Partition the other vertices by the color (say, red or blue) of the edge by which they are connected to v_0 . Choose a second pivot vertex v_1 from the larger of these two sets, and ‘discard’ the vertices from the smaller set. Repeat the process with v_1 and the remaining vertices. When all of the vertices are accounted for, there ought to be at least $1 + \log_2(4^n) = 2n + 1$ pivots. To every pivot v_i we may assign a color such that, $\forall j > i$, the edge between v_i and v_j is of that color. Now at least half of the vertices are of one color; the induced subgraph on $n+1$ of those vertices is a homogeneous K_{n+1} .

Fill in the details.

Exercise 4 (Erdős, 1949). Prove $2^{n/2} \not\rightarrow (n+1, n+1)$.

Hint: Apply the probabilistic method to show there is a counterexample.

Exercise⁺ 5. $16 \not\rightarrow (3, 3, 3)$. Exhibit a 3-coloring of the edges of K_{16} with no homogeneous subgraph isomorphic to K_3 .

Hint: Consider the cosets of the cyclic subgroup of order 5 in \mathbb{F}_{16}^\times . (These are “cyclotomic classes.” See definition 11.)

Exercise 6. Prove

$$[e \cdot k!] \rightarrow \underbrace{(3, 3, \dots, 3)}_k.$$

Hint: We have shown the result for $k = 2, 3$. Apply induction.

2. FERMAT’S LAST THEOREM OVER \mathbb{F}_q

Exercise 7. Show that we may classify all integer solutions to the equation $x^2 + y^2 = z^2$ as triples of the form $d(u^2 - v^2)$, $d(2uv)$, $d(u^2 + v^2)$ for all $d, u, v \in \mathbb{Z}$.

Theorem 8 (Schur). If $q > q_0(k)$ (q_0 a sufficiently large function of k) then

$$x^k + y^k = z^k$$

has a non-trivial solution in \mathbb{F}_q .

Exercise 9. Let $A_k = \{a^k : a \in \mathbb{F}_q^\times\}$. Show $A_k = A_d$, where $d = \gcd(k, q-1)$.

Exercise 10. Prove that the index of A_k in \mathbb{F}_q^\times , $|\mathbb{F}_q^\times : A_k| = k$, assuming $k|(q-1)$.

Definition 11 (Cyclotomic Classes). The cosets of A_k in \mathbb{F}_q^\times are called **cyclotomic classes** of A_k . If we define the homomorphism $\phi : \mathbb{F}_q^\times \rightarrow \mathbb{F}_q^\times$ by

$$\phi(x) = x^{\frac{q-1}{k}},$$

then $A_k = \text{Ker}(\phi)$. Moreover, $\text{Im}(\phi) = A_{\frac{q-1}{k}}$.

Exercise 12. In \mathbb{F}_{13} , find A_3, A_4 , and all corresponding cyclotomic classes.

Exercise 13. In order to prove Schur's theorem, we identify the vertex set of K_q with \mathbb{F}_q . The edge connecting x and y is labeled by $\phi(x-y)$, representing the cyclotomic class of $x-y$. This yields a k -coloring of K_q . If q is sufficiently large, $q > q_0(k)$, then there exists a homogeneous K_3 subgraph on some vertices $x, y, z \in \mathbb{F}_q^\times$. Recall the bound we proved earlier,

$$q_0(k) = \lceil ek! \rceil.$$

Thus,

$$\phi(x-y) = \phi(y-z) = \phi(x-z) = a,$$

hence $(x-y), (y-z), (x-z) \in A_k b$, where $\phi(b) = a$. Write

$$\begin{aligned} x-y &= x_1^k b \\ y-z &= y_1^k b \\ x-z &= z_1^k b; \end{aligned}$$

now, $(x-y) + (y-z) = (x-z)$, so $x_1^k a + y_1^k a = z_1^k a$. It follows that

$$x_1^k + y_1^k = z_1^k$$

is a solution.

In the above proof, we specified a coloring of K_q using $\phi(x-y)$. However, this approach assumes an orientation on K_q : $x \rightarrow y$. For our purposes, why is orientation irrelevant?