DISCRETE MATHEMATICS PROBLEMS. JUNE 26, 2002

INSTRUCTOR: LÁSZLÓ BABAI

Exercise 1. Prove the Cauchy-Schwarz inequality for a Hermitian inner product: $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Definition 2. A transformation A of a complex vector space V equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$ is *unitary* if $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

Exercise 3. Show that if ϕ is a unitary transformation and λ is an eigenvalue of ϕ, then $|\lambda| = 1$.

Exercise 4. An $n \times n$ complex matrix A is unitary $\Leftrightarrow AA^* = I \Leftrightarrow A^*A = I$, where A^* is the conjugate transpose of A.

Definition 5. A complex matrix A is normal if $AA^* = A^*A$.

Exercise 6. An $n \times n$ matrix A is Hermitian $\Leftrightarrow$ A is normal and all eigenvalues are real.

Exercise 7. An $n \times n$ matrix A is unitary $\Leftrightarrow$ A is normal and all eigenvalues of A have absolute value 1.

Exercise$^+$ 8. (Spectral Theorem) Prove: A is normal if and only if there exists a unitary S such that $S^{-1}AS$ is a diagonal matrix. Equivalently, A is normal if and only if there exists an orthonormal eigenbasis for A.

Exercise 9. Show that if $\chi : G \to \mathbb{C}^\times$ is a character of a finite abelian group G, then $\chi(1_G) = 1 \in \mathbb{C}^\times$.

Exercise 10. Again, G is a finite abelian group. Find the natural isomorphism between G and G^{**}.

Exercise 11. Find an ordering of the characters and elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ so that the character table is the Sylvester matrix S_2.

Exercise 12. Show that the character table of $\mathbb{Z}_2^k$ is the Sylvester matrix S_k.

Notation: If $n \to \underbrace{(m, \cdots, m)}_{k \text{ times}}$, we use the shorthand $n \to (m)_k$.

Exercise 13. Prove that $2^k \not\to (3)_k$. (*Hint.* If $n \not\to (3)_k$, then $2n \not\to (3)_{k+1}$.)

Exercise 14. Prove that $2^{4k} \not\to (3)_{3k}$. (*Hint.* We already know $16 \not\to (3)_3$. Show that if $n \not\to (3)_k$ and $m \not\to (3)_\ell$ then $mn \not\to (3)_{k+\ell}$.)