DISCRETE MATHEMATICS PROBLEMS. JUNE 28, 2002

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`http://people.cs.uchicago.edu/~laci/reu02.dir`

Definition 1. Let G be a finite abelian group, and let $X \subseteq G$. We define

$$\Phi(X) = \max_{\chi \neq \chi_0} \left| \widehat{f_X}(\chi) \right| = \max_{\chi \neq \chi_0} \left| \sum_{g \in X} \chi(g) \right|,$$

where the maximum is taken over all non-principal characters $\chi \in \widehat{G} \setminus \{\chi_0\}$.
Recall that f_X denotes the characteristic function of X:

$$f_X(y) = \begin{cases} 1 & y \in X \\ 0 & y \notin X. \end{cases}$$

Notation 2. Let $X \subseteq G$. We will denote its complement by $\overline{X} = G \setminus X$. The set $-X$ is defined by $-X = \{-x : x \in X\}$. Let $a \in G$. The set $X + a$ is defined by $X + a = \{x + a : x \in X\}$. Let $k \in \mathbb{Z}$, $x \in G$. The product kx is defined by

$$kx = \begin{cases} x + \cdots + x \ (k \text{ times}) & \text{if } k > 0 \\ 0 \ (\text{in } G) & \text{if } k = 0 \\ -x + \cdots + -x \ (-k \text{ times}) & \text{if } k < 0. \end{cases}$$

The set kX is defined by $kX = \{kx : x \in X\}$. $\mathrm{Aut}(G)$ is the group of automorphisms of G, i. e., the group of $G \to G$ isomorphisms.

Exercise 3. Given $X \subseteq G$, and $|G| = n$, prove that

$$\begin{aligned} &\Phi(X) = \Phi(\overline{X}) \\ &\Phi(X) = \Phi(X + a) \qquad a \in G \\ &\Phi(X) = \Phi(-X) \\ &\Phi(X) = \Phi(kX) \qquad k \in \mathbb{Z}, \text{ and } \gcd(k, n) = 1 \\ &\Phi(X) = \Phi(\alpha(X)) \qquad \text{for } \alpha \in \mathrm{Aut}(G). \end{aligned}$$

Exercise 4. Prove

$$|\mathrm{Aut}(\mathbb{Z}_m)| = \varphi(m)$$

where φ is Euler's phi function.

Exercise 5. (Sets with largest non-principal Fourier coefficient)
Let $a_{n,k} = \max\{\Phi(X) : X \subseteq \mathbb{Z}_n, |X| = k\}$. Let $a_n = \max_k a_{n,k}$.

(a) Prove: if n is a prime and $\Phi(X) = a_{n,k}$, where $X \subseteq \mathbb{Z}_n$ and $|X| = k$, then X is an arithmetic progression (mod n) of length k.
(b) Prove: (a) is false if n is composite. *Hint:* let k=2.
(c) Prove: a_p is an increasing function of p for primes p.

(OVER)

(d) Prove: $a_p \sim p/\pi$ for primes p as $p \to \infty$.

(e) Prove the following unusual characterization of π:

$$\pi = \sup_G \min_X \frac{|G|}{\Phi(X)},$$

where G runs over all finite abelian groups, and X runs over all subsets of G.