Linear algebra and applications to graphs
Part 1

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June 17, 2001

1 Basic Linear Algebra

Exercise 1.1 Let V and W be linear subspaces of $\mathbb{F}^n$, where $\mathbb{F}$ is a field, $\dim V = k$, $\dim W = \ell$. Show that $\dim(V \cap W) \geq n - (k + \ell)$.

Exercise 1.2 Let A be an $n \times n$ matrix over the field $\mathbb{F}$ and $\mathbf{x} \in \mathbb{F} \setminus \{0\}$. Then $(\exists \mathbf{x})(A\mathbf{x} = 0) \Leftrightarrow \det(A) = 0$, where $\det(A)$ is the determinant of A.

Definition 1.3 Let A be an $n \times n$ matrix over the field $\mathbb{F}$ and $\mathbf{x} \in \mathbb{F}^n \setminus \{0\}$. We say that $\mathbf{x}$ is an **eigenvector** for A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda \mathbf{x}.$$

Exercise 1.4 Show that if $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{F}^n$ are eigenvectors with distinct eigenvalues then they are linearly independent.

Definition 1.5 The **characteristic polynomial** of the $n \times n$ matrix A is

$$f_A(x) := \det(xI - A).$$

Exercise 1.6 λ is an eigenvalue of A if and only if it is a root of $f_A(x)$, i.e. $f_A(\lambda) = 0$.

Exercise 1.7 Let $f_A(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ (why is it monic?). Show that $a_0 = (-1)^n \det(A)$ and $a_{n-1} = -\operatorname{tr}(A)$, where the **trace** of A is defined as $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$ (sum of the diagonal elements).

Definition 1.8 If λ is an eigenvalue of A then the **geometric multiplicity** of λ is $\dim\ker(A - \lambda I)$ (the number of linearly independent eigenvectors for eigenvalue λ). The **algebraic multiplicity** of λ is its multiplicity as a root of $f_A(x)$.

CONVENTION. By the multiplicity of the eigenvalue (without adjective) we always mean the **algebraic multiplicity**.

Exercise 1.9 The algebraic multiplicity of λ is greater than or equal to its geometric multiplicity.

Exercise 1.10 If A is an $n \times n$ matrix then the algebraic multiplicity of the eigenvalue λ equals $\dim\ker(A - \lambda I)^n$.

Definition 1.11 The $n \times n$ matrices A and B are **similar**, $A \sim B$, if there exists an invertible matrix S s.t. $A = S^{-1}BS$.

Exercise 1.12 Show that if A and B are similar then $f_A(x) = f_B(x)$.

Definition 1.13 An **eigenbasis** for A is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

Definition 1.14 A is **diagonalizable** if it is similar to a diagonal matrix.

Exercise 1.15 A is diagonalizable if and only if it has an eigenbasis.

Exercise 1.16 If A is an upper triangular matrix then its eigenvalues, with proper algebraic multiplicity, are its diagonal elements.

Exercise 1.17 Every matrix over $\mathbb{C}$ is similar to an upper triangular matrix. More generally, a matrix over the field $\mathbb{F}$ is similar to a triangular matrix if and only if all roots of f_A belong to $\mathbb{F}$.

Exercise 1.18 Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix A (listed with their algebraic multiplicities). Then $\det(A) = \prod_i \lambda_i$ and $\operatorname{tr}(A) = \sum_i \lambda_i$.

Exercise 1.19 Show that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not similar to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Verify that their characteristic polynomials are identical. Show that the second matrix is not diagonalizable.

Exercise 1.20 Let $\mathbb{F} = \mathbb{C}$ (or any algebraically closed field). Show that A is diagonalizable if and only if each eigenvalue of A have same geometric and algebraic multiplicities.

Exercise 1.21 If A is a diagonal matrix, $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, and f is a polynomial then $f(A) = \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}$.

Definition 1.22 $m_A(x)$, the **minimal polynomial** of A, is the monic polynomial of lowest degree, such that $m_A(A) = 0$.

Exercise 1.23 Show that $m_A(x)$ exists and $\deg m_A \leq n^2$.

Exercise 1.24 Show that if $f \in \mathbb{F}[x]$ is a polynomial then $(f(A) = 0) \Leftrightarrow (m_A \mid f)$.

Theorem 1.25 (Cayley-Hamilton Theorem)

$$m_A | f_A \quad \textit{or, equivalently,} \quad f_A(A) = 0.$$

Consequently $\deg m_A \leq n$.

Exercise 1.26 A proof of the Cayley-Hamilton theorem over $\mathbb{C}$ is outlined in this series of exercises:

1. Prove Cayley-Hamilton for diagonal matrices.
2. Prove the theorem for diagonalizable matrices.
3. Show that if A_i is a sequence of matrices, $\lim_{i\to\infty} A_i = A$, and f_i is a sequence of polynomials of the same degree, and $\lim_{i\to\infty} f_i = f$ (coefficientwise convergence) then $\lim_{i\to\infty} f_i(A_i) = f(A)$. In other words, polynomials of matrices are continuous functions of the matrix entries and the coefficients of the polynomials.

4. Show that for any matrix A there exists a sequence of diagonalizable matrices A_i, such that $\lim_{i\to\infty} A_i = A$. In other words diagonalizable matrices form a dense subset of the set of all matrices.
(Hint: prove it first for upper triangular matrices.)

5. Complete the proof of Cayley-Hamilton theorem over $\mathbb{C}$.

Exercise 1.27 Complete the proof of the Cayley-Hamilton Theorem (over any field) by observing that if an identity of (multivariate) polynomials holds over $\mathbb{Z}$ then it holds over any commutative ring with identity.

2 Euclidean Spaces

In this section the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $a \in \mathbb{C}$, we will denote by $\bar{a}$ the complex conjugate of a. Of course, if $a \in \mathbb{R}$, then $\bar{a} = a$. Similarly if A is a matrix then each entry of $\bar{A}$ is the complex conjugate of the corresponding entry of A.

Definition 2.1 Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$. Their (standard) **inner product** is

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum i = 1^n \bar{x}_i y_i.$$

Definition 2.2 If $\mathbf{x} \in \mathbb{F}^n$ is a vector, its **norm** is defined as

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Definition 2.3 If A is a matrix, then the **adjoint** matrix is $A^* = \bar{A}^T$ (conjugate-transpose).

Exercise 2.4 We think of vectors as column matrices. Verify the following:

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y}$

2. $\langle A^* \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle$

3. $((\forall \mathbf{x}, \mathbf{y})(\langle B\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle)) \Leftrightarrow B = A^*$

Exercise 2.5 $(AB)^* = B^* A^*$, where A,B are not necessarily square matrices. (What dimensions should they have so that we can multiply them?)

Definition 2.6 We say that a matrix A is **self-adjoint** if $A^* = A$. A is also called **symmetric** if $\mathbb{F} = \mathbb{R}$ and **Hermitian** if $\mathbb{F} = \mathbb{C}$.

Exercise 2.7 If $A = A^*$ then all eigenvalues of A are real.

Exercise 2.8 Show that the characteristic polynomial of a Hermitian matrix has real coefficients.

Definition 2.9 The **quadratic form** associated with a self-adjoint matrix A is a function $Q_A(\mathbf{x}) : \mathbb{F}^n \to \mathbb{F}$, defined by

$$Q_A(\mathbf{x}) = \mathbf{x}^* A\mathbf{x} = \sum_{i,j} a_{ij}\bar{x}_i x_j.$$

Definition 2.10 The **operator norm** of a matrix A is defined as

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Exercise 2.11 Show that $\|A\| = \sqrt{\lambda_1(A^*A)}$, where $\lambda_1(A^*A)$ denotes the largest eigenvalue of A^*A. (Note that A^*A is self-adjoint. A does not need to be a square matrix for this exercise.)

Definition 2.12 A self-adjoint matrix A is called **positive semidefinite** if $(\forall \mathbf{x} \in \mathbb{F})(Q_A(\mathbf{x}) \geq 0)$. A is called **positive definite** if $(\forall \mathbf{x} \in \mathbb{F} \setminus \{0\})(Q_A(\mathbf{x}) > 0)$.

Exercise 2.13 Show that a self-adjoint matrix is positive definite (resp. semidefinite) if and only if all its eigenvalues are positive (resp. nonnegative).

Exercise 2.14 Show that a self-adjoint matrix A is positive definite if and only if all of its upper left corner determinants $\det(a_{11})$, $\det \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, etc, are positive.
(Hint: Use the Interlacing Theorem (given in Exercise 2.29 below).)

Definition 2.15 A set of vectors $\mathbf{a_1}, \ldots, \mathbf{a_n}$ is **orthonormal** if $\langle \mathbf{a_i}, \mathbf{a_j} \rangle = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

Definition 2.16 A is a **unitary** matrix if $A^*A = I$.

Exercise 2.17 Show that if A is unitary and λ is an eigenvalue of A then $|\lambda| = 1$.

Exercise 2.18 Let A be an $n \times n$ matrix. Prove that the following are equivalent:

1. A is unitary;
2. $AA^* = I$;
3. the columns of A form an orthonormal basis of $\mathbb{F}^n$;
4. the rows of A form an orthonormal basis of $\mathbb{F}^n$.

Definition 2.19 The $n \times n$ matrix A is **normal** if $AA^* = A^*A$.

Exercise$^+$ 2.20 Show that A is normal if and only if A has an orthonormal eigenbasis. Equivalently, show that A is normal if and only if there exists a unitary matrix S such that S^*AS is a diagonal matrix. If so, the entries in the diagonal are the eigenvalues of A.

Exercise 2.21 Show that the $n \times n$ matrix A is

1. self-adjoint if and only if A is normal and all its eigenvalues are real;
2. unitary if and only if A is normal and all its eigenvalues have unit absolute value.

NOTATION: For the rest of this section we use the following notation: $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$; A is a self-adjoint matrix over $\mathbb{F}$. The eigenvalues of A (with multiplicity) are $\lambda_1 \geq \ldots \geq \lambda_n$.

Now we will formulate the fundamental

Theorem 2.22 (Spectral Theorem) *The eigenvalues of A are real and A has an orthonormal eigenbasis.*

Let $\mathbf{e_1}, \ldots, \mathbf{e_n}$ be an orthonormal eigenbasis for A.

Exercise 2.23 Show that if $\mathbf{x} = \sum_i x_i \mathbf{e_i}$, then

$$Q_A(\mathbf{x}) = \sum_i \lambda_i |\mathbf{x}_i|^2.$$

Definition 2.24 The **Rayleigh quotient** is the function $R(x) : \mathbb{F}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$R(\mathbf{x}) = \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \frac{Q_A(\mathbf{x})}{\|\mathbf{x}\|^2}.$$

Exercise 2.25 Show that

$$\lambda_1 = \max_{\|\mathbf{x}\|=1} R(\mathbf{x}).$$

Exercise 2.26 Show that

$$\lambda_2 = \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{e_1}}} R(\mathbf{x});$$

$$\lambda_3 = \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{e_1},\mathbf{e_2}}} R(\mathbf{x});$$

and so on.

Exercise 2.27 Show that if $\lambda_1 = \mathbf{x}^* A\mathbf{x}$, $\|\mathbf{x}\| = 1$, then $A\mathbf{x} = \lambda_1\mathbf{x}$.

Exercise 2.28 (Fischer-Courant Theorem)

$$\lambda_i = \max_{\substack{U\leq F^n \\ \dim U=i}} \min_{x\in U} R(x)$$

where the maximum runs over all linear subspaces $U \leq \mathbb{F}^n$ of dimension i.

Exercise 2.29 (Interlacing Theorem) Let A be an $n\times n$ self-adjoint matrix. We can construct a new $(n-1)\times(n-1)$ matrix by removing the ith row and the ith column of A. The resulting matrix B is self-adjoint. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A and $\mu_1 \geq \ldots \geq \mu_{n-1}$ be the eigenvalues of B (with multiplicity). Show that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n$.

3 Applications to Graph Theory

There are two important square matrices commonly associated to graphs – the adjacency matrix of the graph, and the (finite or combinatorial) Laplacian. This allows us to apply the theory of eigenvalues to graphs, and it turns out that a great deal of information about the graph is carried in the spectra of these matrices.

For graph theory terminology please refer to “Graph Theory Terminology“ handout.

3.1 The Adjacency Matrix

Definition 3.1 Let $G = (V, E)$ be a graph; assume $V = [n] = \{1, 2, \ldots, n\}$. The **adjacency matrix** $A_G = (a_{ij})$ of G is the $n \times n$ $(0, 1)$-matrix defined by $a_{ij} = 1$ if $\{i, j\} \in E$ (vertices i and j are adjacent); and $a_{ij} = 0$ otherwise. Note that $a_{ii} = 0$.

Exercise 3.2 Show that the (i, j) entry of $(A_G)^k$ gives the number of walks of length k between vertex i and vertex j. Give an interpretation for the (i, i) entry of $(A_G)^k$ and for $\sum_{j=1}^{n}(A_G)_{ij}$.

The adjacency matrix acts on functions on the graph. That is, if $f : V \to \mathbb{R}$ is a function on the vertices of the graph (which can also be considered a column matrix), then

$$Af(i) = \sum_{\substack{j \\ \{i,j\} \in E}} f(j).$$

Notice that this action is just matrix multiplication.

Exercise 3.3 Isomorphic graphs have similar adjacency matrices.

This allows us to make the following definitions:

Definition 3.4 κ is an **eigenvalue** of G if it is an eigenvalue of A_G. The **characteristic polynomial** of G is the characteristic polynomial of A_G. The **spectrum** of G is the ordered set of all eigenvalues of A_G (with multiplicities).

As before we will assume that eigenvalues of G are always ordered $\kappa_1 \geq \ldots \geq \kappa_n$.

Exercise 3.5 Compute the spectrum of each of the following graphs: K_n (the complete graph on n vertices), the star on n vertices (a tree with a vertex of degree $n-1$, denoted $K_{n-1,1}$), $K_{k,\ell}$ (the complete bipartite graph).

Exercise 3.6 Let $G = (V, E)$ be a graph. Let G_i be the graph obtained by deleting the ith vertex (and the edges incident with it) from G. Show that eigenvalues of G and G_i interlace.

Exercise 3.7 $(\forall i)$ $(|\kappa_i| \leq \kappa_1)$.

Exercise 3.8 If G is connected then $\kappa_1 > \kappa_2$.

Exercise 3.9 If G is bipartite , then $\kappa_{n-i} = -\kappa_{i+1}$.

Exercise 3.10 If G is connected and $\kappa_1 = -\kappa_n$ than G is bipartite. Thus if G is connected and not bipartite then $(\forall i > 1)$ $(|\kappa_i| < \kappa_1)$.

Exercise 3.11 $\kappa_1 \leq \max_i \deg_G(i)$.

Exercise 3.12 $\kappa_1 \geq \frac{2|E|}{n} = \frac{1}{n}\sum_i \deg_G(i)$ (average degree).

Exercise 3.13 If G is k-regular, i.e. $(\forall i)(\deg_G(i) = k)$, then $\kappa_1 = k$.

Exercise 3.14 If $\kappa_1 = \max_i \deg_G(i)$ and G is connected then G is regular.

Exercise 3.15 If $\kappa_1 = \frac{1}{n}\sum_i \deg_G(i)$ then G is regular.

Exercise 3.16

1. Upper bounds on the maximal eigenvalue are hereditary; that is, if $H \subset G$ is a subgraph, then $\kappa_1(H) \leq \kappa_1(G)$.
2. Show that upper bounds on the second eigenvalue κ_2 fail to be hereditary in general, but are hereditary in the special case that H is an *induced* subgraph.
   (Hint: for the first part, consider the spectrum of K_n. For the second part, recall the Interlacing Theorem.)

Exercise 3.17 If $\mathrm{diam}(G) = d$, then the number of distinct eigenvalues of A_G is at least $d + 1$.
(Hint: Prove that under the diameter hypothesis, $I, A, \ldots, A^d$ are linearly independent. To show this, recall the significance of the (i, j) entry of A^k from Exercise 3.2.)

3.2 The Laplacian and Expansion of a Graph

Definition 3.18 We define the **Laplacian** of the graph G to be

$$\Delta_G = D_G - A_G$$

where A_G is the adjacency matrix and D_G is a diagonal matrix, $D_G(i, i) = \deg_G(i)$.

Exercise 3.19 Verify that for $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^T$,

$$\mathbf{x}^* \Delta_G \mathbf{x} = \sum_{\{i,j\} \in E} |x_i - x_j|^2.$$

Exercise 3.20 Show that Δ_G is positive semidefinite.

However, Δ_G is *not* positive definite:

Exercise 3.21 Check that $\Delta_G \mathbf{j} = 0$, where $\mathbf{j} = (1, \ldots, 1)^T$.

Exercise 3.22 Show that if G is connected, then 0 is a simple eigenvalue.

Exercise 3.23 Prove that the multiplicity of 0 as an eigenvalue of Δ_G is equal to the number of connected components of G.

Therefore if $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are eigenvalues of Δ_G then $\lambda_2 = 0$ if and only if G is disconnected.

Definition 3.24 (Fiedler) λ_2 is the **algebraic connectivity** of G.

Exercise 3.25 If G is a k-regular graph (every vertex has degree k) and $k = \kappa_1 \geq \ldots \geq \kappa_n$ are eigenvalues of A_G and $0 = \lambda_1 \leq \ldots \leq \lambda_n$ are eigenvalues of Δ_G the $\lambda_i + \kappa_i = k$. In particular, $\lambda_2 = \kappa_1 - \kappa_2$. λ_2 is also referred to as the **eigenvalue gap** or **spectral gap**.

Definition 3.26 If $A \subseteq G$ we denote by $\delta(A)$ the number of edges between A and $\overline{A} = V \setminus A$. The **isoperimetric ratio** for A is $\frac{\delta(A)}{|A|}$. The **isoperimetric constant** of G is

$$i_G = \min_{\substack{A \neq \emptyset \\ |A| \leq \frac{n}{2}}} \frac{\delta(A)}{|A|}.$$

The next result shows the important fact that if λ_2 is large then G is "highly expanding".

Exercise* 3.27 $\lambda_2 \leq 2\,\frac{\delta(A)}{|A|}$.

Later we will state a companion result which shows that in some sense if λ_2 is small then G has a small isoperimetric constant.