

Character sums, Weil's Estimates and Paradoxical Tournaments

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1 Characters of finite fields

Definition 1.1 A *character* of a finite field F is a function $\chi: F \rightarrow \mathbb{C}$, satisfying the following conditions:

1. $\chi(0) = 0$
2. $\chi(1) = 1$
3. $(\forall a, b \in F)(\chi(ab) = \chi(a)\chi(b))$.

Note that a character is a homomorphism from the multiplicative group $F^\times = F \setminus \{0\}$ to the multiplicative group \mathbb{C}^\times .

Example 1.2 For any field F , we define the *principal character*, χ_0 , by $\chi_0(0) = 0$ and $(\forall a \neq 0)(\chi_0(a) = 1)$.

Notation. For a prime power $q = p^k$, \mathbb{F}_q denotes the field of order q (i. e., the field \mathbb{F}_q has q elements). For $k = 1$, the field \mathbb{F}_p is the field of mod p residue classes. Note that for $k \geq 2$, the mod p^k residue classes do *not* form a field, so for $k \geq 2$, the field \mathbb{F}_q is not the same as the ring of residue classes mod q . It is known, however, that for every prime power q there exists a field \mathbb{F}_q and this field is unique up to isomorphism. If you are not familiar with finite fields, you may still read this note, always replacing q by p .

Example 1.3 When $F = \mathbb{F}_p$ for an odd prime p , we define the *quadratic character* $\chi(a) := \left(\frac{a}{p}\right)$, where $\left(\frac{a}{p}\right)$ is 0 when $a = 0$, 1 when a is a quadratic residue, and -1 when a is a quadratic nonresidue. $\left(\frac{a}{p}\right)$ is called the *Legendre symbol*.

Exercise 1.4 Show that, for all a , $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$.

Next, we extend the concept of the **quadratic character** to all finite fields of odd order.

Example 1.5 Let \mathbb{F}_q be a finite field of odd order q . The *quadratic character* χ of \mathbb{F}_q is defined as follows: for $a \in \mathbb{F}_q$,

$$\chi(a) = \begin{cases} 1 & \text{if } (\exists b \in \mathbb{F}_q)(a = b^2 \neq 0); \\ -1 & \text{if } (\forall b \in \mathbb{F}_q)(a \neq b^2); \\ 0 & \text{if } a = 0. \end{cases}$$

Exercise 1.6 Let q be an odd prime power and χ the quadratic character of \mathbb{F}_q . Prove: if $q \equiv -1 \pmod{4}$ then $\chi(-1) = -1$; and if $q \equiv 1 \pmod{4}$ then $\chi(-1) = 1$.

Exercise 1.7 For any prime power q , prove: $(\forall a \in \mathbb{F}_q)(a^{q-1} = 1)$.

(Note that for $q = p$ a prime, this is Fermat's Little Theorem.) *Hint.* Use Lagrange's theorem from group theory (the order of a subgroup divides the order of the group).

The *order* of a nonzero element $a \in \mathbb{F}_q$ is the smallest positive k such that $a^k = 1$. It follows from the preceding exercise that $k \mid q - 1$ (" k divides $q - 1$ ").

Corollary 1.8 $(\forall a \in \mathbb{F}_q)(\chi(a) \text{ is a complex root of unity})$.

Indeed, if $a^k = 1$ then $(\chi(a))^k = \chi(a^k) = \chi(1) = 1$.

Definition 1.9 The *order* of a character is the least positive integer s such that $\chi(a)^s = 1$ for all $a \in F$, $a \neq 0$.

Note that, for any character of \mathbb{F}_q , the order s must divide $q - 1$.

The following is a basic fact about the structure of finite fields.

Theorem 1.10 *For any prime power q , the multiplicative group \mathbb{F}_q^\times is cyclic. Equivalently, there exists some $g \in \mathbb{F}_q^\times$ such that $\mathbb{F}_q^\times = \{g, g^2, \dots, g^{q-1} = 1\}$.*

Such an element g is called a *generator* of \mathbb{F}_q^\times , or a *primitive root* of the field \mathbb{F}_q .

Exercise 1.11 Prove the Theorem. *Hint.* Use Sylow's Theorem from group theory and the fact that a polynomial of degree n has at most n roots in a field.

Corollary 1.12 *If χ is a character of \mathbb{F}_q of order s , and g is a primitive root of \mathbb{F}_q , then $\chi(g)$ is a primitive s^{th} root of unity. Conversely, for any $\omega \in \mathbb{C}$ such that $\omega^{q-1} = 1$, there exists a unique character χ of \mathbb{F}_q with $\chi(g) = \omega$.*

Exercise 1.13 Prove the Corollary.

Note that if we take $\omega = 1$ we get the principal character, and, for q odd, if we take $\omega = -1$, we get the quadratic character.

2 Character Sum: Weil's Theorem

In this section we describe one of the most beautiful results of 20th century mathematics.

First we consider the sum of characters over all elements of a field.

Exercise 2.1 If $\chi \neq \chi_0$, then $\sum_{a \in \mathbb{F}_q} \chi(a) = 0$.

Let now f be a polynomial of degree d over \mathbb{F}_q . We wish to estimate the sum

$$S(\chi, f) = \sum_{a \in \mathbb{F}_q} \chi(f(a))$$

Clearly, since $|\chi(f(a))|$ is 0 or 1 for all a , we have $|S(\chi, f)| \leq q$. This is the best possible upper bound; for example, if f is identically 1 then $S(\chi, f) = q$; if χ is the quadratic character and $f(x) = x^2$, then $S(\chi, f) = q - 1$.

Amazingly, once the trivial exceptions have been eliminated, a much stronger bound holds on the magnitude of $S(\chi, f)$: the values of the character tend to cancel each other out roughly by the same amount as if they were chosen to be ± 1 by coin flips.

Theorem 2.2 (André Weil) *Let \mathbb{F}_q be a finite field, and let χ be a character of \mathbb{F}_q of order s . Let $f(x)$ be a polynomial of degree d over \mathbb{F}_q such that $f(x)$ cannot be written in the form $c(h(x))^s$, where $c \in \mathbb{F}_q$. Then*

$$\left| \sum_{a \in \mathbb{F}_q} \chi(f(a)) \right| \leq (d-1)\sqrt{q}.$$

Thus, in a sense, the values of a character over the range of a polynomial behave as “random” values, even though they are fully “deterministic.” This feature is the key to a large number of applications to combinatorics and the theory of computing where the goal is “derandomization”: the elimination of random choice from the proof of existence of a combinatorial object, i. e., replacing a probabilistic proof of existence by an explicit construction.

3 k -paradoxical tournaments: a proof by the Probabilistic Method

Let $X = (V, E)$ be a digraph. Let $x \in V$ and $A \subseteq V$. We say that x **dominates** A if $(\forall a \in A)((x, a) \in E)$. We write $x \rightarrow A$ to denote this statement.

Definition 3.1 A digraph $X = (V, E)$ is **k -paradoxical** if $(\forall A \subset V)(|A| = k \Rightarrow \exists x \in V)(x \rightarrow A)$.

Definition 3.2 A *tournament* is a digraph $T = (V, E)$ in which for every pair $\{x, y\}$ of vertices, exactly one of the following holds: $x = y$ or $(x, y) \in E$ or $(y, x) \in E$.

Note that this concept corresponds to diagrams of round-robin tournaments without draws and without rematches. An edge (arrow) from a to b indicates that player a beat player b .

In a 1-paradoxical tournament, every player is beaten by someone. In a 2-paradoxical tournament, every pair of players is beaten by someone. Even 2-paradoxical tournaments are not straightforward to construct.

Exercise 3.3 Construct a 2-paradoxical tournament on 7 players. *Hint.* Make your diagram have a symmetry of order 7.

So it is quite surprising that k paradoxical tournaments actually do exist for every k . Constructing such tournaments even for $k = 3$ is quite hard. However, Paul Erdős, in one of the gems of his Probabilistic Method, demonstrated the *existence* of such tournaments without telling us how to construct them.

Theorem 3.4 (Erdős) *If $n > ck^22^k$ then there exists a k -paradoxical tournament on n vertices. (c is an absolute constant.)*

What Erdős has shown is not just that such tournaments *exist*, but they *abound*: almost every tournament on a given set of n vertices (players) is k -paradoxical. The model of “random tournaments” is very simple: flip a coin to decide the outcome of each match.

Exercise 3.5 Let $A \subset V$ be a set of k players (out of the set V of n players) and let x be a player, not in A . Calculate the probability that $x \rightarrow A$.

Exercise 3.6 Let A be as before. Show that the probability that none of the remaining $n - k$ players dominates A is exactly $(1 - 2^{-k})^{n-k}$.

Exercise 3.7 Infer from the preceding exercise that the probability that our random tournament is not k -paradoxical is less than

$$\binom{n}{k} (1 - 2^{-k})^{n-k}. \quad (1)$$

Exercise 3.8 Conclude that if $\binom{n}{k} (1 - 2^{-k})^{n-k} \leq 1$ then there exists a k -paradoxical tournament on n vertices.

Exercise 3.9 Prove that if $k \geq 3$ and $n > 4k^22^k$ then the inequality in the preceding exercise will hold. (A constant $c > 4$ works for $k = 2$; smaller constants work for larger values of k . As $k \rightarrow \infty$, the value of a suitable constant $\rightarrow 1$.) *Hint.* Use the following facts: $\binom{n}{k} < n^k/k!$; $1 - x < e^{-x}$; and the monotonicity of the function $x/\ln x$.

This concludes the proof of Erdős's Theorem.

Exercise 3.10 Prove that if $n > ck^22^k$ (for some absolute constant c) then almost all tournaments on a given set of n players are k -paradoxical.

Here "almost all" means that for every $\epsilon > 0$ there exists n_0 such that if $n > n_0$ and $n > ck^22^k$ then the probability that the random tournament is k -paradoxical is greater than $1 - \epsilon$. *Hint.* Revisit the same calculations done for the previous exercises. Only minimal modifications are needed.

4 Paley tournaments: an explicit construction of k -paradoxical tournaments

We describe an explicit construction of k -paradoxical tournaments for arbitrarily large k .

Definition 4.1 Let $q \equiv -1 \pmod{4}$ be a prime power and let χ denote the quadratic character of \mathbb{F}_q . The *Paley tournament of order q* is defined as a digraph $P(q) = (V, E)$ where $V = \mathbb{F}_q$; we have a directed edge $a \rightarrow b$ iff $\chi(a - b) = 1$.

Note that because $q \equiv -1 \pmod{4}$, we have $\chi(-1) = -1$ (Exercise 1.6). Since the character is multiplicative, this ensures that $\chi(a - b) = -\chi(b - a)$, so there is exactly one directed edge between any two distinct vertices. This shows that $P(q)$ is a tournament. (We also need to note that $\chi(0) = 0$, so there are no loops in the digraph.)

Theorem 4.2 (Graham-Spencer) If $q \equiv -1 \pmod{4}$ and $q \geq k^24^k$, then $P(q)$ is a k -paradoxical tournament.

Proof: Let $A = \{a_1, \dots, a_k\} \subset V$ be an arbitrary k -subset. Let $N = \#\{x \in V : x \rightarrow A\}$ be the number of vertices which dominate the set A . We seek to show that $N > 0$. In fact, we will show that $N \approx \frac{q}{2^k}$.

Consider the following three cases:

- $x \rightarrow A \Rightarrow (\forall i)(\chi(x - a_i) = 1)$.
- $x \not\rightarrow A$ and $x \notin A \Rightarrow (\forall i)(\chi(x - a_i) = \pm 1)$ and $(\exists i)(\chi(x - a_i) = -1)$.
- $x \in A \Rightarrow (\exists i)(\chi(x - a_i) = 0)$.

Now let $\psi(x) := \prod_{i=1}^k (\chi(x - a_i) + 1)$. Considering the cases above, we have

$$\psi(x) = \begin{cases} 2^k, & x \rightarrow A \\ 0, & x \not\rightarrow A, x \notin A \\ 0 \text{ or } 2^{k-1}, & x \in A \end{cases}$$

The case $\psi(x) = 2^{k-1}$ occurs for at most one $x \in A$; namely, if and only if x dominates the rest of A .

Thus, we can compute the sum $S := \sum_{x \in \mathbb{F}_q} \psi(x) = 2^k N + \epsilon 2^{k-1}$, where $\epsilon \in \{0, 1\}$. We will have succeeded in showing that $N > 0$ if we can prove that S is large ($S > 2^{k-1}$ will suffice).

Using the notation $[k] := \{1, \dots, k\}$, we obtain the expansion

$$S = \sum_{x \in \mathbb{F}_q} \prod_{i=1}^k (\chi(x - a_i) + 1) = \sum_{x \in \mathbb{F}_q} \sum_{I \subseteq [k]} \prod_{i \in I} \chi(x - a_i).$$

Letting $f_I(x) := \prod_{i \in I} (x - a_i)$ and using the multiplicativity of χ we see that

$$S = \sum_{x \in \mathbb{F}_q} \sum_{I \subseteq [k]} \chi(f_I(x)) = \sum_{I \subseteq [k]} \sum_{x \in \mathbb{F}_q} \chi(f_I(x)) = \sum_{x \in \mathbb{F}_q} \chi(f_\emptyset(x)) + \sum_{I \neq \emptyset} \sum_{x \in \mathbb{F}_q} \chi(f_I(x)).$$

Let us denote by R the rest of the sum: $R := \sum_{I \neq \emptyset} \sum_{x \in \mathbb{F}_q} \chi(f_I(x))$. Since the empty product is 1 and $\chi(1) = 1$, we have $S = (\sum_{\mathbb{F}_q} 1) + R = q + R$. If we can show that q dominates R then we shall be done since then $N \approx S/2^k \approx q/2^k$, as desired. Now

$$|R| = \left| \sum_{I \neq \emptyset} \sum_{x \in \mathbb{F}_q} \chi(f_I(x)) \right| \leq \sum_{I \neq \emptyset} \left| \sum_{x \in \mathbb{F}_q} \chi(f_I(x)) \right| \leq \sum_{I \neq \emptyset} (|I| - 1) \sqrt{q} \quad (\text{by Weil}).$$

Note we can apply Weil because f_I , by definition, has no multiple roots, so in particular $f_I(x) \neq c(h(x))^2$. There are 2^k choices for $I \subseteq [k]$ and, for each choice, $|I| \leq k$. Thus, we have shown that $|R| < 2^k \cdot (k - 1) \sqrt{q}$.

From above, $S = 2^k N + \epsilon 2^{k-1} = q + R$, so

$$N > \frac{q}{2^k} - (k - 1) \sqrt{q} - \frac{1}{2} > \frac{q}{2^k} - k \sqrt{q}.$$

So for $N > 0$ it suffices that $\frac{q}{2^k} \geq k \sqrt{q}$, i. e., $q \geq k^2 4^k$. □