Discrete Math, Second series, First Problem Set (July 18)

REU 2003

Instructor: László Babai

Definition 0.1. A **product-free set** in a group G is a subset $L \subset G$ such that the equation $xy = z$ has no solution in L.

Definition 0.2. A **triangle-free set** in a group G is a subset $T \subset G$ such that the equation $xyz = 1$, $x, y, z \in G$ implies $x = y = z$.

Exercise 0.3. Prove that every abelian group of order $n \geq 2$ has a product-free subset of size $\geq n/3$. Prove that this statement remains valid for solvable grops.

Exercise 0.4. Prove that $G = S_n$ (the symmetric group of degree n) has a product-free subgroup of size $|G|/2$.

Exercise 0.5. Prove that A_n (the alternating group of degree n) has a product-free subset of size $|G|/n$.

Exercise 0.6. Prove that A_4 has a product-free subset of size $|G|/3$.

Exercise$^+$ 0.7. Open questions (a) For $n \geq 5$, does $G = A_n$ have a product-free subset of size $1 + |G|/n$? (b) If a_n denotes the size of the largest product-free subset of A_n, does $a_n/|A_n|$ go to zero?

Exercise$^+$ 0.8. Analyse the previous questions regarding triagle-free sets in the place of product-free sets. See that difficulties arise already for certain abelian groups.

Exercise 0.9. Prove that the automorphism group of the cube is isomorphic to $S_4 \times Z_2$.

Exercise 0.10. Prove that the automorphism group of the dodecahedron is isomorphic to $A_5 \times Z_2$.

Exercise 0.11. Prove that the automorphism group of the Petersen graph is isomorphic to S_5.