Discrete Math, Second Series, 10th Problem Set (August 8)

REU 2003

Instructor: László Babai
Scribe: Ivona Bezakova

0.1 Revisiting Hamidoune's Theorem

Exercise$^+$ 0.1. Find a vertex transitive digraph without positive atoms or prove that this cannot happen. (The instructor does not know the answer.)

0.2 Growth rate of groups: Gromov's Theorem

We can prove Gromov's Lemma (see Exercise 0.19 in the 9th Problem Set handout) using sphere packing and covering arguments (see Hint). It is the first lemma in the proof of a major result by M. Gromov.

Definition 0.2. Let G be a group. $G' = \langle [a,b] : a,b \in G \rangle$ denotes the *commutator subgroup*, i.e. subgroup generated by all commutators.

Exercise 0.3. $G' = \{1\}$ iff G is abelian.

Exercise 0.4. Let $N \lhd G$. Prove: G/N is abelian if and only if $N \geq G'$.

Definition 0.5. For $K, L \leq G$ we write $[K, L] := \langle [a,b] : a \in K, b \in L \rangle$.

Notice that $G' = [G, G]$.

Definition 0.6. We define the *lower central series of* G inductively: $K^0(G) := G$ and $K^{l+1}(G) := [G, K^l(G)]$ for $l \geq 0$.

Notice that $K^1(G) = G'$.

Definition 0.7. A group G is *nilpotent* if $(\exists l)(K^l(G) = \{1\})$.

Exercise 0.8. Prove: $(\forall G)(\forall i)(K^i(G) \lhd G)$.

Exercise 0.9. Prove that $K^i(G)/K^{i+1}(G)$ is abelian. In fact, $K^i(G)/K^{i+1}(G) \leq Z(G/K^{i+1}(G))$.

Definition 0.10. The *derived chain* of G is defined inductively: $D^0(G) := G$ and $D^{i+1}(G) = (D^i(G))'$. The group G is *solvable* if $(\exists i)(D^i(G) = \{1\})$.

Notice that $D^1(G) = G'$.

Exercise 0.11. Prove that for all i, $D^i(G) \lhd G$.

Exercise 0.12. Prove: if G is nilpotent then G is solvable.

Exercise 0.13. Prove: if $G' \leq Z(G)$ then G is nilpotent.

Exercise 0.14. Prove: $S'_n = A_n$.

Exercise 0.15. D_n is nilpotent iff n is a power of 2.

Exercise 0.16. Prove: $D'_n = \begin{cases} \mathbb{Z}_{n/2} & n \text{ even} \\ \mathbb{Z}_n & n \text{ odd} \end{cases}$

Exercise 0.17. If G is a finite p-group (i.e. the order of every element is a power of p) then $Z(G) \neq \{1\}$.

Corollary 0.18. *All finite p-groups are nilpotent.*

Definition 0.19. If G is an (infinite) abelian group, we define the *torsion subgroup* of G as $T(G) = \{g \in G : \text{ord}(g) < \infty\}$.

If G is a finitely generated abelian group then $G = T \times H$ where $H \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$.

Definition 0.20. Functions $f(n), g(n) \to \infty$ have the same *rate of growth* if $(\exists k)((\forall n)(f(kn) > g(n))$ and $(g(kn) > f(n)))$. In this case, we will write $f(n) \approx g(n)$.

Exercise 0.21. All quadratic functions have the same rate of growth, e. g., $3n^2 - 3n + 5 \approx 100n^2$.

Notice that the Θ relation and the rate of growth are different notions: $e^n \approx e^{2n}$ but $e^n \neq \Theta(e^{2n})$. More generally, $(\forall a, b > 1)(a^n \approx b^n)$.

For a vertex-transitive infinite, locally finite graph X we want to know the rate of growth of $f(n) = |B(x, n)|$ where $B(n, x)$ denotes the ball of radius n about vertex x. Let $X = \Gamma(G, S)$ where S is a finite set of generators satisfying $S = S^{-1}$ (i. e., X is undirected).

Exercise 0.22. Every Cayley graph of $G = \mathbb{Z} \times \mathbb{Z}$ has quadratic rate of growth.

Exercise 0.23. The rate of growth depends on G only, not on the generating set S chosen. We shall use $f_G(n)$ to denote a representative of this equivalence class.

Remark 0.24. An example of an infinite, locally finite graph with exponential rate of growth is a regular tree of degree 3. (This is a Cayley graph of the free product of three cyclic groups of order 2, i.e. $\langle a, b, c \colon\ a^2 = b^2 = c^2 = 1\rangle$.) The free group F_2 on two generators a, b has exponential rate of growth since $\Gamma(F_2, \{a, b, a^{-1}, b^{-1}\})$ is a regular tree of degree 4.

Theorem 0.25 (J. Milnor, J. A. Wolf). *Finitely generated nilpotent groups have polynomial rate of growth.*

Exercise 0.26. If G is finitely generated and $H \leq G$ with $|G : H|$ finite then

1. H is also finitely generated [Schreier],
2. $f_G(n) \approx f_H(n)$ (G and H have the same rate of growth) [Milnor].

Definition 0.27. A group is *virtually nilpotent* if it has a nilpotent subgroup of finite index.

Corollary 0.28. *Finitely generated virtually nilpotent groups have polynomial rate of growth.*

Theorem 0.29 (Milnor, Wolf 1968). *If G is finitely generated and solvable then either G isvirtually nilpotent or G has exponential rate of growth.*

Theorem 0.30 (Jacques Tits, 1972). *If $G \leq GL(n, \mathbb{R})$ is a finitely generated linear group then either G is virtually nilpotent or G has exponential rate of growth.*

The following major result settles the question of polynomial growth, raised by Milnor in connection with his study of the curvature of differentiable manifolds.

Theorem 0.31 (Gromov, 1981). *G has polynomial rate of growth iff G is virtually nilpotent.*

This result immediately raised the question of the existence of finitely generated groups of intermediate growth rates. The existence of such groups was shows by Grigorchuk.

Theorem 0.32 (Grigorchuk, 1983). *There exist groups of growth rate $2^{\sqrt{n}} < f(n) < 2^{n^{0.98}}$, i.e. intermediate growth rate.*

OPEN PROBLEM: Do there exist groups with rate of growth superpolynomial but less than $2^{\sqrt{n}}$?