Discrete Math, Second Series, 4th Problem Set (July 25)

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We focus our attention on automorphism groups of regular solids. A tetrahedron is just another way of representing the complete graph on 4 vertices i.e., K_4. The automorphism group $\mathrm{Aut}(K_4) = S_4$. More generally, we have $\mathrm{Aut}(K_n) = S_n$, where K_n is the complete graph on n vertices.

Next, we consider a cube. We may label the vertices of the cube so that the four in the top face are $1, 2, 3$ and 4, while the diagonally opposite ones in the bottom face are $1', 2', 3'$ and $4'$ respectively. $\mathrm{Aut}(\text{cube}) = S_4 \times \mathbb{Z}_2$. Here the nonidentity element of $\mathbb{Z}_2$ corresponds to the central involution (i.e., reflection about the center of the cube) which is orientation reversing, while elements of S_4 correspond to the orientation preserving automorphisms of the cube.

Exercise 0.1. If $A \in M_3(\mathbb{R})$, then A has a real eigenvector. ($M_n(\mathbb{R})$ is the set of all $n \times n$ matrices with entries from $\mathbb{R}$.)

Exercise 0.2. In $\mathbb{R}^3$, every orientation preserving congruence which fixes the origin is a rotation. (Note: a linear transformation A is *orientation preserving* if $\det(A) > 0$.

By considering the action on pairs of opposite vertices of the cube (the main diagonals of the cube) we obtain the map $\mathrm{Aut}(\text{cube}) \to S_4$. The kernel of this map is <central reflection>. The image is all of S_4, so the map is onto. To see this, we define ρ as a rotation about the vertical axis of the cube, and $\tilde{\rho}$ as the corresponding element of S_4 induced by ρ. Then ρ permutes the main diagonals so that $\tilde{\rho}(\{i, i'\}) = (\{i+1, (i+1)'\}) \bmod 4$. Similarly, if we let σ be rotation about the diagonal $(1, 1')$ by $120°$, then the induced permutation $\tilde{\sigma}$ permutes the other three diagonals. The next exercise completes the argument.

Exercise 0.3. $S_4 =< \tilde{\rho}, \tilde{\sigma} >$.

Exercise 0.4. If C is any centrally symmetric bounded subset of $\mathbb{R}^3$, then the group of congruences of C, $\mathrm{Aut}(C) = \mathbb{Z}_2 \times \mathrm{Aut}^+(C)$. Here $\mathrm{Aut}^+(C)$ is the orientation preserving subgroup.

Exercise 0.5. Show that the above statement fails to hold when C is not bounded. To see this, construct a central symmetric subset C of $\mathbb{R}$ such that the group of congruences of S has trivial center. *Hint.* Either $C = \mathbb{R}$ or $C = \mathbb{Z}$ will do. Show that if ρ_i denotes the reflection about point $i \in \mathbb{R}$, and τ the translation by 1 to the right, then $(\rho_0)^{\tau^i} = (\tau^i)^{-1}(\rho_0)(\tau^i) = \rho_i$.

Exercise 0.6. Find a finite group G such that (a) the center of G is neither the identity nor G; (b) $(\nexists\, A \le G)(G = Z(G) \times A)$. Find the smallest such group G.

Definition 0.7. The **semidirect product** of groups K and L with respect to $\alpha : K \to \mathrm{Aut}(L)$ is the group $G = L \rtimes_\alpha K$ defined as the set $\{(l,k)\colon l \in L, k \in K\}$ with the operation $(l,k)(l',k') = (l l'^{\alpha(k)}, kk')$.

Exercise 0.8. With the above definition, show that (a) $(l,k) = (l,1)(1,k)$; (b) $(1,k)(l,1)(1,k)^{-1} = (l^{\alpha(k)},1)$; (c) $L \cong \{(l,1)\colon l \in L\} \lhd G$; and (d) $K \cong \{(1,k)\colon k \in K\} \le G$.

Definition 0.9. Let $G \le S_k$ and H be any abstract group. The **wreath product** of H by G is defined to be $H \wr G = H^k \rtimes_\alpha G$ where $\alpha : G \to \mathrm{Aut}(H^k)$ is the permutation action of G on the k components of H^k.

Let $X = X_i \cup \cdots \cup X_k$ be a graph with connected components X_i and $X_i \cong X_j$ for all $1 \le i,j \le k$. Then $\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \wr S_k$.

If $G \le S_k$ and $H \le \mathrm{Sym}(\Omega)$, then the wreath product $H \wr G$ naturally acts on $\Omega_1 \dot\cup \cdots \dot\cup \Omega_k$ ($|\Omega_i| = |\Omega|$). Here H_i, the ith component of H^k acts on Ω_i, and G permutes the H_i. In this case, $H \wr G \le S_{kl}$ (where l is the degree of $\mathrm{Sym}(\Omega)$). This is called the "imprimitive representation" of $H \wr G$.

Remark 0.10. $H^k \lhd H \wr G$, $G \le H \wr G$, and $(H \wr G)/H^k \cong G$.

Let Q_n be the graph of the n-cube. So $|Q_n| = 2^n$, and we may think of the vertices of this graph as elements of $\{0,1\}^n$, i. e., strings of length n consisting of 0s and 1s. We define the **Hamming distance** between two strings of equal length to be the number of places where they differ. Two vertices in Q_n will be adjacent if their Hamming distance is 1.

Switching coordinates of vertices independently corresponds to reflections about various planes which shows that $(\mathbb{Z}_2)^n \le \mathrm{Aut} Q_n$. Similarly, permuting the positions of coordinates corresponds to rotations about different lines, which gives us $S_n \le \mathrm{Aut}(Q_n)$.

Exercise 0.11. $\mathrm{Aut}(Q_n) = \mathbb{Z}_2 \wr S_n$.

Remark 0.12. $(\mathbb{Z}_2)^n \lhd \mathrm{Aut}(Q_n)$ and $\mathrm{Aut}(Q_n)/(\mathbb{Z}_2)^n \cong S_n$.

The "primitive representation" of $H \wr G$ is the action of $H \wr G$ on $\Omega_1 \times \cdots \times \Omega_k = \Omega^k = \{(a_1, \ldots, a_k)\colon a_i \in \Omega\}$ given by $(a_1, \ldots, a_k)^{(h_1,\ldots,h_k;g)} = (a^{h_1}_{1^{g^{-1}}}, \ldots, a^{h_k}_{k^{g^{-1}}})$.

Corollary 0.13. $S_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \wr S_3$.

The octahedron is the dual of the cube. Hence its automorphism group is the same as that of the cube.

Exercise 0.14. Aut(Octahedron) $\cong$Aut(Cube).

Exercise 0.15. Generalize the above exercise to n-dimensions.

$\mathrm{Aut}(\mathrm{Dodecahedron}) = A_5 \times \mathbb{Z}_2$. Here, as before, the element in $\mathbb{Z}_2$ is the central reflection and A_5 is the subgroup of the orientation preserving automorphisms.

Exercise 0.16. Show that the action of the rotation group of the dodecahedron on the 5 triples of perpendicular axes connecting opposite edges is A_5.

Exercise 0.17. The action on the 10 pairs of opposite vertices in the dodecahedron is primitive.

Exercise 0.18. Show that the automorphism group of Petersen's graph (see handout) is S_5. To do this, observe that Petersen's graph may be constructed by defining the vertex set to be $\binom{5}{2}$ i.e., all pairs of 5 objects, and adjacency to be given by disjointness (so that two pairs are adjacent if they are completely disjoint).