Discrete Math, Second series, 7th Problem Set (August 1)

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0.1 Set theory

Definition 0.1. S is a **countably infinite** set if there exists a 1-1 correspondence between S and $\mathbb{N}$.

Exercise 0.2. Prove that the following sets are countably infinite: $\mathbb{Q}$, $\mathbb{Z}^n$, $\mathbb{Q}^n$, the set of algebraic numbers.

Definition 0.3. A set S has the cardinality of **continuum** if there exists a 1-1 correspondence between S and $[0, 1]$.

Definition 0.4. Let S be a set. The **power set** of S, denoted 2^S, is the set of all subsets of S.

Exercise 0.5. Prove that the following sets have the cardinality of continuum: $\mathbb{R}$, $\mathbb{R}^n$, $C[0, 1]$ (set of all continuous functions defined on $[0, 1]$), $\mathrm{Sym}(\mathbb{Z})$ (set of all permutations of $\mathbb{Z}$), $2^{\mathbb{Z}}$.

Definition 0.6. We say that the cardinality of a set T is at least as big as the cardinality of a set S, i.e. $\mathrm{card}(S) \leq \mathrm{card}(T)$, if there exists an injective function from S to T.

Exercise 0.7. Prove: If $\mathrm{card}(S) \leq \mathrm{card}(T)$ and $\mathrm{card}(T) \leq \mathrm{card}(S)$ then $\mathrm{card}(S) = \mathrm{card}(T)$, i.e. there exists a bijection between S and T.

Theorem 0.8 (Cantor). $\mathrm{card}(2^S) > \mathrm{card}(S)$*, i. e., there is no 1-1 correspondence between* S *and its power set* 2^S*.*

0.2 Continuum Hypothesis and the Axiom of Choice

Definition 0.9. The **continuum hypothesis** (CH) says that there is no cardinality strictly between countable and continuum. The **generalized continuum hypothesis** (GCH) says that for all infinite sets S there is no cardinality strictly between $\mathrm{card}(S)$ and $\mathrm{card}(2^S)$.

Definition 0.10. The **Axiom of Choice** states that, given any family $\{A_i : i \in I\}$ of nonempty sets, there exists a function f with domain I such that $(\forall i \in I)(f(i) \in A_i)$.

In the 1930s Gödel showed that no contradiction arises if the GCH is added to ZFC, where ZF stands for the Zermelo-Fraenkel axioms of set theory and C stands for the Axiom of Choice. This work was complemented by Paul Cohen who proved in the 19060s that no contradiction arises if the negation of CH is added to ZFC. Therefore the Continuum Hypothesis is **independent** of ZFC.

Another interesting result of Cohen states that the Axiom of Choice is independent of ZF.

In this class we always assume the Axiom of Choice.

0.3 Cardinal numbers

Definition 0.11. A set S is said to be **well-ordered** if every nonempty subset of S has a minimum.

Theorem 0.12 (Well-ordering Theorem: Cantor). *Every set can be well-ordered.*

The Well-ordering Theorem is equivalent to the Axiom of Choice. It is immediate that the Axiom of Choice follows from the Well-Ordeing Theorem (why?). For the proof of the converse, we recommend Van der Waerden's (Modern) Algebra.

Corollary 0.13. *Cardinal numbers are well-ordered.*

Following Cantor, the inventor of inifinite cardinal numbers, we use the notation $\aleph_\alpha$ to denote infinite cardinalities ($\aleph$ is "aleph," the first letter of the Hebrew alphabet). $\aleph_0$ is the countable cardinality; $\aleph_1$ is the smallest uncountable cardinality; $\aleph_2$ is next, etc. The CH states that $\aleph_1$ is the continuum.

According to Cohen, not only is it consistent with ZFC that continuum is greater than $\aleph_1$, it is also consistent that continuum is $\aleph_\alpha$ where α is an ordinal number of arbitrarily large cardinality less than continuum.

Theorem 0.14 (Cantor). *If A is an infinite set then* $\text{card}(A \times A) = \text{card}(A)$.

This theorem is equivalent to the Axiom of Choice.

Exercise 0.15. Prove: For cardinalities a, b the following holds: $a + b = ab = \max\{a, b\}$.

Exercise 0.16. Prove: *continuum* $\times$ *continuum* $=$ *continuum*, i.e. $\text{card}([0,1] \times [0,1]) = \text{card}([0,1])$.

Exercise 0.17. Find a continuous function from $[0,1]$ onto $[0,1]^2$.

0.4 Chromatic number of infinite graphs

Definition 0.18. The chromatic number $\chi(X)$ of an infinite graph X is defined as the minimum cardinality of a set of colors required for a legal coloring.

Theorem 0.19 (Erdős, Hajnal). *If X is not countably colorable, i. e., $\chi(X) > \aleph_0$, then X contains a complete bipartite subgraph $K_{m,\aleph_1}$ for each positive integer m. In particular, X contains a 4-cycle.*

Exercise 0.20. Construct a graph H such that $(\forall m)(H \supseteq K_{m,\aleph_1})$ but $H \not\supseteq K_{\aleph_0,\aleph_1}$.

Exercise 0.21. If S is a minimal set of generators of G (i. e., no subset of S generates G) then $\chi(\Gamma(G,S)) \leq \aleph_0$.

Exercise 0.22. Construct a countable abelian group which has no minimal set of generators. *Hint.* One of your most standard abelian groups will do.

Exercise 0.23. Every group has a countably colorable Cayley-graph. *Hint.* Read on before solving.

Exercise 0.24. Prove that every group has a *sequentially nonredundant* sequence of generators, i. e., a well-ordered sequence of generators $\{g_\beta \ : \ \beta < \alpha\}$ such that for all β, g_β is not generated by $\{g_\gamma \ : \ \gamma < \beta\}$.

Exercise 0.25. Prove that if S is a sequentialy nonredundant sequence of generators, for the group G then the Cayley graph $\Gamma(G,S)$ does not contain the complete bipartite graph $K_{5,17}$.

0.5 Ramsey Theory

Definition 0.26. We write $a \to (b,c)$ if any graph on a vertices contains a clique of order b or an independent set of order c.

We can view this definition as follows. If $a \to (b,c)$ then no matter how we color the edges of K_a by two colors there always will be a monochromatic subgraph K_b colored by the first color or a monochromatic subgraph K_c colored by the second color.

Exercise 0.27. Prove: $\aleph_0 \to (\aleph_0, \aleph_0)$.

Surprisingly, one can two-color the edges of $K_{\mathbb{R}}$ (the complete graph on the real numbers) such that every monochromatic complete subgraph will be countable, i.e. $\aleph_1 \not\to (\aleph_1, \aleph_1)$.

Theorem 0.28 (Sierpinszki). *continuum* $\not\to (\aleph_1, \aleph_1)$.

Exercise 0.29. If $S \subseteq \mathbb{R}$ is a well-ordered set then $|S| \leq \aleph_0$.

Definition 0.30. Cardinal a is **weakly-inaccessible** if

(1) a is a limit cardinal, i.e. it has no immediate predecessor,

(2) a cannot be written as a sum of fewer smaller cardinals.

The examples of limit cardinals include $\aleph_0$ and $\aleph_\omega := \sup\{\aleph_n : n \text{ an integer}\}$.

Definition 0.31. Cardinal a is **strongly-inaccessible** if

(1') $2^b < a$ for all $b < a$,

(2) as above.

By definition continuum is not strongly-inaccessible but could be weakly-inaccessible.

Exercise 0.32. Prove: If $\kappa \to (\kappa, \kappa)$ then κ is weakly-inaccessible.

Exercise 0.33. If $\kappa \to (\kappa, \kappa)$ then κ is strongly-inaccessible. *Hint.* Sierpinszki's Theorem.