Discrete Math, Second series, 8th Problem Set (August 4)

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Let X be a directed graph. Recall that $\kappa(X)$ is the vertex connectivity of X, i.e. the maximal k such that for any two distinct vertices $u, v \in V(X)$ there are k vertex disjoint paths between u and v.

Let $F \subseteq V(X)$. Let $X^+(F) \subseteq V(X) \setminus F$ be the set of out-neighbors of F. Note that by definition $F \cap X^+(F) = \emptyset$. The set $X^-(F)$ is defined analogously (in-neighbors).

Exercise 0.1. If $F \cup X^+(F) \neq V(X)$ then $X^+(F) \geq \kappa(X)$. Analogously, if $F \cup X^-(F) \neq V(X)$ then $X^-(F) \geq \kappa(X)$.

Definition 0.2. A set $F \subseteq V(X)$ is called a *positive fragment* if $F \cup X^+(F) \neq V(X)$ and $X^+(F) = \kappa(X)$. Negative fragments are defined analogously, using $X^-(F)$.

Definition 0.3. Let $a(X)$ denote the minimum size of all fragments (positive and negative) of the digraph X. A fragment (positive or negative) of size $a(F)$ is called an *atom*. If an atom is a positive fragment, it is called a *positive atom*. Negative atoms are defined analogously.

Exercise 0.4. Construct a strongly connected digraph which has no negative atoms.

Exercise 0.5. Show that the directed cycle has $\kappa = 1$ and all atoms have size 1. What are the positive fragments in the directed cycle?

Exercise 0.6. Construct a graph with $\kappa = 1$ in which all atoms have size 100.

Exercise 0.7. Construct a graph in which the union of atoms does not cover the entire graph.

Exercise$^+$ 0.8. (Hamidoune's Theorem) If F is a positive fragment and A is a positive atom then either $A \cap F = \emptyset$ or $A \subseteq F$.

Exercise 0.9. Assume X is a vertex-transitive digraph (i.e., $\mathrm{Aut}(X)$ acts transitively on $V(X)$). Assume also that positive atoms exist. Then the positive atoms partition the vertex set.

Exercise 0.10. Suppose that X is vertex-primitive (i.e. $\mathrm{Aut}(X)$ acts as a primitive group on $V(X)$ (no non-trivial invariant partition)). Show that each atom is a single vertex. Conclude that $\kappa(X)$ is equal to the outdegree of vertices in X.

Exercise 0.11. Prove the Hamidoune, Mader-Watkins connectivity results.

Recall that the Cauchy-Davenport Theorem states that if $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ then $|A + B| \geq \min\{|A| + |B| - 1, p\}$.

Exercise 0.12. Prove Cauchy-Davenport Theorem using Exercise 0.10.

A Hamilton cycle in an undirected graph is a cycle that passes through all vertices. We say that a graph X is Hamiltonian if it has a Hamiltonian cycle.

Exercise 0.13. Show that the dodecahedron is Hamiltonian.

Given a graph X let $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ be the eigenvalues of its adjacency matrix.

Exercise 0.14. Show that the eigenvalues of the cycle with n vertices are $2\cos(2\pi\lfloor i/2 \rfloor)$ for $i = 1, \ldots, n$.

Exercise 0.15. Show that for symmetric matrices A, B, $\lambda_i(A + B) \geq \lambda_i(A) + \lambda_n(B)$.

Exercise 0.16. Show that if a cubic graph X is Hamiltonian then $\lambda_i(X) \geq 2\cos(2\pi\lfloor i/2 \rfloor) - 1$ for $i = 1, \ldots, n$.

Exercise 0.17. Show that the eigenvalues of the Petersen's graph are $3, 1, 1, 1, 1, 1, -2, -2, -2, -2$.

Exercise 0.18. Infer from the preceding sequence of exercises that the Petersen's graph is not Hamiltonian. (This proof is due to B. Mohar.)

Coxeter's graph is obtained from the Fano plane as follows. It has a vertex for each triangle (i. e. a triple of points which do not lie on a line). Two vertices are connected if the corresponding triangles are disjoint. Note that Coxeter's graph has 28 vertices. It is known that Coxeter's graph is not Hamiltonian.

The only two other known non-Hamiltonian vertex transitive graphs are obtained from Petersen's graph and Coxeter's graph by replacing each vertex by a 3-cycle.

Observation 0.19. Removing $k \geq 1$ vertices from a cycle results in at most k connected components.

Definition 0.20. A graph X is tough is for any k removal of k vertices results in $\leq k$ connected components.

A Hamilton path in an undirected graph X is a path that passes through all vertices.

Exercise 0.21. Show that 4×4 chessboard with edges corresponding to knight's moves does not have a Hamilton path. Give an "ah-ha" proof. *Hint.* Show that for some k one can remove k vertices from this graph such that the rest falls into $k + 2$ connected components. Show that this precludes a Hamilton path.

Exercise 0.22. Show that if k, ℓ are odd then $k \times \ell$ grid is not tough.

Exercise 0.23. All vertex transitive graphs are tough. (HINT: Use the Mader-Watkins edge connectivity result.)

Exercise 0.24. If X is a connected vertex transitive graph then every pair of longest cycles shares at least 1 vertex. In fact they share at least 3 vertices.

Exercise 0.25. Let $G \leq \mathrm{Sym}([n])$ be a transitive permutation group and let $A, B \subseteq [n]$. Suppose that for all $g \in G$, $|A \cap B^g| \geq r$. Then

$$|A| \cdot |B| \geq nr.$$

Exercise 0.26. Show that the longest cycle in a connected vertex transitive graph has length $\geq \sqrt{3n}$.