Discrete Math, Second series, 9th Problem Set (August 6)

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Recall that $\chi(X)$ is the chromatic number of X and $\alpha(X)$ is the independence number of X (size of the largest independent set). (An *independent set,* or anticlique, is a set of pairwise non-adjacent vertices).

Exercise 0.1. Show that $\alpha(X)\chi(X) \geq n$.

Exercise 0.2. Show that $\chi(X)$ is not bounded above by any function of $n/\alpha(X)$.

Exercise 0.3. Prove: $\alpha(X) + \chi(X) \leq n + 1$.

Exercise 0.4. Prove: $\alpha(X)\chi(X) \leq (n+1)^2/4$. *Hint.* Use the preceding exercise and the inequality between the geometric and arithmetic means.

Exercise 0.5. Prove that the bounds in the preceding two exercises are tight for all odd n.

This preceding exercise shows that $\chi(X)$ can be much larger (by a factor of $\Omega(n)$) than its lower bound $n/\alpha(X)$, so this lower bound is far from being tight. Contrast this with the situation for vertex-transitive graphs:

Exercise 0.6. If X is vertex-transitive then we have nearly matching lower and upper bounds for $\chi(X)$ in terms of n and $\alpha(X)$: $\chi(X) \leq \frac{n(1+\ln n)}{\alpha(X)}$.

Definition 0.7. A sequence $a_1, \ldots, a_n$ is *unimodal* if there is k such that $a_1, \ldots, a_k$ is increasing and $a_k, \ldots, a_n$ is decreasing (not necessarily strictly). A sequence $a_1, \ldots, a_n$ is *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for all i.

Exercise 0.8. If a sequence is log-concave then it is unimodal.

Exercise 0.9. Prove that the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.

Definition 0.10. A graph X is *distance-transitive* if $\forall a, b, x, y \in V(X)$ if $\operatorname{dist}(a,b) = \operatorname{dist}(x,y)$ then $(\exists g \in \operatorname{Aut} X)(a^g = x, b^g = y)$.

Exercise 0.11. Construct infinitely many connected graphs that are vertex-transitive but not distance transitive.

Exercise 0.12. Show that Petersen's graph is distance-transitive.

Kneser's graph $K(n, s)$, $n \geq 2s+1$ has $\binom{n}{s}$ vertices corresponding to the subsets of $[n]$ of size s with two vertices being adjacent if the corresponding sets are disjoint. *Johnson's graph* $J(n, s)$, $n \geq s+1$ has $\binom{n}{s}$ vertices corresponding to the subsets of $[n]$ of size s with two vertices being adjacent if the corresponding sets have symmetric difference of size 2.

Exercise 0.13. Show that the n-cube, Kneser's graph $K(n, s)$, Johnson's graph $J(n, s)$ are distance transitive.

Let $S(x, r)$ denote the sphere of radius r about vertex x, i. e.

$$S(x, r) = \{y \in V(X) \mid \operatorname{dist}(x, y) = r\}.$$

Exercise 0.14. Let X be distance-transitive. Let $a_r = |S(x, r)|$ for some $x \in V(X)$. (So $a_0 = 1$.) Show that the sequence $\{a_r\}$ is log-concave.

Exercise 0.15. Construct infinitely many connected vertex-transitive graphs such that the sequence sequence $\{a_r\}$ is not unimodal.

Exercise 0.16. PROJECT. How pathological can the sequence $\{a_r\}$ be for connected vertex-transitive graphs? Is it possible to have a_1 "large," and a_2 "much larger," a_3 "even larger," then a_4 "much smaller" than a_3, and then a_5 much larger than a_4, perhaps much larger even than a_3? What kind of peaks and valleys can the sequence $\{a_r\}$ have? – While all these exercises are for finite graphs, can an infinite vertex-transtive graph have a_1, a_2, a_3 infinite, a_4 finite, and then a_5 infinite again?

Exercise 0.17. If $a_0, a_1, \ldots$ is log-concave then $(\forall i \leq j)(\forall k \geq 1)(a_{k-i}a_{j+k} \leq a_i a_j)$.

Let $B(x, r)$ be the ball of radius r around vertex x, i. e.

$$B(x, r) = \{y \in V(X) \mid \operatorname{dist}(x, y) \leq r\}.$$

Lemma 0.18 (Gromov). *Let X be a vertex-transitive graph. Let $f(r) = |B(x, r)|$ for some x. Then*

$$f(r)f(5r) \leq f(4r)^2.$$

In Gromov's Lemma, X may be infinite but it must be *locally finite* (the vertices have finite degree).

Exercise 0.19. Prove Gromov's Lemma. (*Hint:* Let Y be the maximal set of vertices at pairwise distance $\geq 2r+1$ within $B(3r, x)$. Prove $|Y| \cdot f(r) \leq f(4r)$ and $|Y| \cdot f(4r) \geq f(5r)$.)