Discrete Math, First Problem Set (June 23)

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Example

$$\pi = 3 + (\pi - 3) \tag{1}$$

$$\pi = 3 + \frac{1}{\frac{1}{\pi - 3}} \tag{2}$$

$$\pi = 3 + \frac{1}{7 + \frac{1}{15{,}99659}} \tag{3}$$

Continuing in this manner we can approximate π as by a continued fraction. Terminating the continued fraction after k iterations gives us an ordinary fraction, the **"k^{th} convergent."** For example the 3^{rd} convergent of π is

$$3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} = \frac{g_3}{h_3}$$

Let now $\alpha \in \mathbb{R}$ and let g_k/h_k be the k^{th} convergent of α.

Exercise 1. Prove that the convergents approximate α from alternating sides.

Exercise 2. Show $g_k h_{k+1} - g_{k+1} h_k = \pm 1$.

Exercise 3. Show that for all $\alpha \in \mathbb{R}$,

$$\left| \alpha - \frac{g_n}{h_n} \right| < \frac{1}{h_n h_{n+1}},$$

where the fractions f_n/g_n are the convergents of α. Infer that the limit of the convergents of α is $\lim_{k \to \infty} \frac{g_k}{h_k} = \alpha$.

Exercise 4. (Quadratic approximability) Prove that for all $\alpha \in \mathbb{R}$ there exist infinitely many fractions p/q such that $|\alpha - p/q| < 1/q^2$.

Exercise 5. Show h_k increases exponentially in k.

Definition 1. A subset S of $[0, 1]$ has (Lebesgue) **measure zero** if for all $\epsilon > 0$, the set S can be covered by (infinitely many) intervals of total length less than ϵ.

Definition 2. Let us say that α is "good" if there exists $\epsilon > 0$ such that there exist infinitely many fractions $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

The next exercise will show that quadratic approximation is the best we can get for most real numbers.

Exercise 6. (Most numbers cannot be super-quadratically approximated) Show that almost all numbers are bad. (I. e., show that the set of good numbers has measure zero.)

Definition 3. An **algebraic number** is an element of $\mathbb{C}$ which is a zero of a (not identically zero) polynomial with integer coefficients. The **degree** of the algebraic number α is the smallest degree of a polynomial with integer coefficients of which α is a zero. A **transcendental number** is a number that is not algebraic.

Exercise 7. (Liouville's Theorem). If α is an irrational algebraic number of degree n, then $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{n+1}}$ has only a finite number of solutions (p, q).

Exercise 8. (Liouville) Prove that there exists an irrational number α such that there exist infinitely many solutions (p, q) to the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{q^q}}.$$

Such an α is a "Liouville number." Prove that there are continuum many Liouville numbers.

Exercise 9. Prove: all Liouville numbers are transcendental.

This will complete the proof of Liouville's celebrated result: the existence of transcendental numbers.

Note that the same proof would work with any super-polynomially growing $f(q)$ in place of q^{q^q}. ($f(q)$ is said to grow **super-polynomially** if $(\forall k)(\exists q_0)(\forall q > q_0)(f(q) > q^k)$.)

Later in the 19th century Cantor gave an alternative proof of the existence of transcendental numbers. In contrast to Liouville, Cantor did not produce any explicit transcendental numbers; yet he proved that the overwhelming majority of real numbers are transcendental, by introducing the hierarchy of infinite of cardinalities.

Exercise 10. (Cantor) Show that the set of algebraic numbers is countable, i. e., it can be put in one-to-one correspondence with the positive integers.

Exercise 11. (Cantor) Show that the set of real numbers is not countable.

The following result is a major improvement over Liouville's Theorem; it netted its author a Fields Medal.

Theorem 1. **(K. F. Roth)** If α is an irrational algebraic number then $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$ has only a finite number of solutions (p, q).

Definition 1. A **monic polynomial** is a polynomial where the leading coefficient is 1. For instance, $x^3 - 3x + 4$ is

Definition 2. An **algebraic integer** is an algebraic number which is a zero of a monic polynomial with integer coefficients. For instance, $sqrt[3]2$ and the Golden Ratio are algebraic integers, corresponding to the monic polynomials $x^3 - 2$ and $x^2 - x - 1$.

Exercise 12. Show that if α is a rational algebraic integer then α is an integer.

Exercise 13. Show: the set of algebraic numbers is a field (it is closed under addition, multiplication, and division).

Exercise 14. * Show: the set of algebraic integers is a ring (it is closed under addition and multiplication).

Exercise 15. Let $\phi = \frac{1+\sqrt{5}}{2}$ (golden ratio). Show that the continued fraction expansion of ϕ is

$$1 + \cfrac{1}{1 + \frac{1}{1+\ldots}}$$

and the convergents of this continued fraction are the quotients of consecutive Fibonacci numbers. Show that $\frac{F_{n+1}}{F_n} \to \phi$.

Exercise 16. Find the continued fraction expansion of $\sqrt{2}$.

Exercise 17. Prove that if a continued fraction is periodic, then its limit α is algebraic of degree 2.

Exercise 18. Find the continued fraction expansion for e.

Theorem 2. **(Dirichlet)(Simultaneous diophantine approximation)** For all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\epsilon > 0$ there exist integers $p_1, \ldots, p_n$ and $q < \frac{1}{\epsilon^n}$ such that

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\epsilon}{q}.$$

Exercise 19. Prove: for all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ there exist infinitely many $(n+1)$-tuples of integers $p_1, \ldots, p_n$ and $q \neq 0$ such that

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+1/n}}.$$

For most n-tuples $(\alpha_1, \ldots, \alpha_n)$ of reals, degree-$(1 + 1/n)$ approximation is the best we can get. Formalize and prove this statement:

Exercise 20. Generalize Exercise 6 to simultaneous approximations.

Linear Algebra review – June 23, 2003

For definitions and examples related to fields and vector spaces, see Chapter 2 (handout).

Exercise 21. Show $\mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$ is a field. What you need to show that this set is closed under taking reciprocals.

Exercise 22. (Fundamental Inequality of Linear Algebra).
If $w_1, \ldots, w_\ell$ are linearly independent vectors in $\operatorname{span}\{v_1, \ldots, v_k\}$ then $\ell \leq k$.

Exercise 23. (Modularity of dimension)

$\dim(U_1 \cap U_2) + \dim(U_1 + U_2) = \dim U_1 + \dim U_2$ for two subspaces U_1 and U_2;

Exercise 24. (Submodularity of rank)

$$\operatorname{rk}(S_1 \cap S_2) + \operatorname{rk}(S_1 \cup S_2) \leq \operatorname{rk}(S_1) + \operatorname{rk}(S_2) \text{ for two subsets.}$$

Exercise 25. Prove: Given a basis $b_1, \ldots, b_n$ for V_1 and arbitrary vectors $w_1, \ldots, w_n \in V_2$ there exists a unique linear map $f : V_1 \to V_2$ such that $f(b_i) = w_i$ for all i.