Discrete Math, Fifth Problem Set (June 27)

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Question: How should we define 0^0?

Trouble: two conventions conflict: $x^0 = 1$ and $0^x = 0$. We shall argue that this conflict can be resolved and that $0^0 = 1$ is the reasonable choice.

Argument 1: **empty products.** The $0^0 = 1$ convention is consistent with the conventions $\sum_{i\in\emptyset} a_i = 0$ and $\prod_{i\in\emptyset} a_i = 1$. (Why are these the only reasonable interpretations of empty sums and products?) The "empty product = 1" rule is used in conventions like $0! = 1$ and $a^0 = 1$.

Argument 2: **combinatorial interpretation of powers.** Let A and B be two finite sets. The number of functions $f : B \to A$ is clearly a^b. Therefore, $0^0 = 1$ (there is only one function $f : \emptyset \to \emptyset$, namely the empty function).

Argument 3: **the limit of x^y.**

Let us consider $\lim_{x,y\to 0^+} x^y$. This limit does not exist. In fact, subsequences can converge to any number between 0 and 1.

Exercise 0.1. Let $0 \le \alpha \le 1$. Prove: there exist sequences $\{x_n\}, \{y_n\}$ of positive numbers such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, and $\lim_{n\to\infty} x_n^{y_n} = \alpha$.

Nonetheless, the limit is "almost well defined."

Exercise 0.2. Prove that $\lim_{x,y\to 0^+} x^y$ equals "mostly" 1. Give a clear definition and prove this statement based on your definition.

Number Theory

Question What is the probability that two random positive integers are relatively prime?

Our first problem is to make sense out of this question. We want every integer to be chosen with equal probability, but then this probability would have to be zero, which is not very helpful.

We need to restrict the domain to a finite segment of the integers and then let the segment grow to infinity.

Example 0.3. Pr(a random integer is divisible by 4) $= \lim_{n\to\infty} \Pr(4 \mid x \colon 1 \le x \le n) = 1/4$

Theorem 0.4 (Prime Number Theorem). *Let $\pi(x)$ be the number of primes $\le x$. Then, $\pi(x) \sim x/\ln x$. ("$\sim$" stands for asymptotic equality, see Handout.)*

Example 0.5. Pr(a random integer is a prime) $= \lim_{n\to\infty} \pi(n)/n \sim \lim_{n\to\infty} 1/\ln n = 0$.

Definition 0.6.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is known that $\zeta(2) = \pi^2/6$ (Euler).

Exercise 0.7. Let $x, y \in \mathbb{N}$ be two integers picked uniformly at random. Assuming the limit $\Pr(\text{g.c.d.}\,(x,y) = 1) := \lim_{n\to\infty} \Pr(\text{g.c.d.}\,(x,y) = 1 \colon 1 \le x, y \le n)$ exists, prove that it must be $1/\zeta(2)$. (Give a three-line proof.)

Definition 0.8. We say that the integers $a_1, \dots, a_k$ are **relatively prime** if $\text{g.c.d.}\,(a_1, \dots, a_k) = 1$. (Note that this does not mean the a_i are pairwise relatively prime; for instance, $6, 10, 15$ are relatively prime.)

Exercise 0.9. Generalize the preceding exercise: If $x_1, \dots, x_k$ are k random integers, then the probability that they are relatively prime is $1/\zeta(k)$.

An Extremal Problem in Discrete Geometry

Definition 0.10. A set $S \subseteq \mathbb{R}^n$ is called a *2-distance set* if there exist $\alpha, \beta \in \mathbb{R}$ such that $(\forall x, y \in S)(x \neq y \Rightarrow \text{dist}(x,y) \in \{\alpha, \beta\})$.

Example 0.11. In the plane a regular pentagon is a 2-distance set. In 3D, an octahedron (dual of the cube) is a 2-distance set.

Exercise 0.12. Prove that the size of a 2-distance set in the plane is at most 5.

Example 0.13. Some examples of 2-distance sets in $\mathbb{R}^n$:

- Set of size n: standard orthonormal basis (vectors $\mathbf{e}_i$),
- Set of size $2n$: hyper-octahedron (vectors $\pm\mathbf{e}_i$),

- Set of size $\binom{n}{2}$: vectors $\mathbf{e}_i + \mathbf{e}_j$ for $i \neq j$.

Exercise 0.14. Find a 2-distance set in $\mathbb{R}^n$ of size $\binom{n+1}{2}$.
Hint: Add a slight idea to the $\binom{n}{2}$ example. Understand it geometrically.

The following theorem shows that this example is asymptotically optimal.

Theorem 0.15 (Larman,Rogers,Seidel). *Let $m(n)$ be the maximum size of a 2-distance set in $\mathbb{R}^n$. Then*

$$m(n) \leq \frac{(n+1)(n+4)}{2}$$

The proof of this theorem is an application of the "linear algebra method:" associate m vectors from some space V with our m objects in such a way that the constraints on our objects imply that the vectors associated will be linearly independent. Then it will follow that $m \leq \dim(V)$. Choose V such that $\dim(V)$ will be the desired bound (in this case, $(n+1)(n+4)/2$).

The trick, of course, is to find the right space V and the way of matching our objects to vectors in V so that the constraints translate into linear independence.

Our space V will be a space of multivariate polynomials; the trick goes back to a paper by Koornwinder.

A complete proof can be found in the "blue book" by Babai and Frankl, page 13.

Extremal Set Theory

Theorem 0.16 (Dijen K. Ray-Chaudhuri, Richard M. Wilson). *Let $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ be a set system satisfying*

1. *uniformity, i.e. $|A_i| = k$ for every i,*
2. *s sizes of intersections, i.e. $|A_i \cap A_j| \in \{\ell_1, \ldots, \ell_s\}$ for every $i \neq j$.*

Then $m \leq \binom{n}{s}$.

Exercise 0.17. In class we proved that $m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$. (This proof can be found in Babai-Frankl, section 5.) Prove the same bound on m without assumption 1.

Remark 0.18. For non-uniform set systems the above bound is tight. Simply take all sets of size at most s.

Exercise 0.19 (R-W Theorem). Under assumptions 1 and 2 prove that $m \leq \binom{n}{s}$.

Hint: Let $\bar{f}_1, \ldots, \bar{f}_m$, be the multi-linear polynomials used in the proof of $m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$. Find $\binom{n}{s-1} + \cdots + \binom{n}{0}$ polynomials independent with the $\bar{f}_i$. More precisely, prove that $\bar{f}_1, \ldots, \bar{f}_n$ and all homogeneous multi-linear monomials of degree less than s (products of at most $s-1$ variables) are linearly independent.

Remark 0.20. The original proofs were more involved and considered matrices of size $\binom{n}{s} \times \binom{n}{s}$. The matrix $M = (m_{i,j})_{m \times n}$, where $m_{i,j} = 1$ if $j \in A_i$ and 0 otherwise, is called an *incidence matrix* of $A_1, \ldots, A_m$.

Ray-Chaudhuri and Wilson used so-called higher incidence matrices for their proof.

Definition 0.21. Inclusion Matrices. The s-inclusion matrix of the set system $A_1, \ldots, A_m \subset [n]$ (where $[n] = \{1, \ldots, n\}$) is an $m \times \binom{n}{s}$ matrix $M_s = (m_{i,j}^{(s)})$ is defined as follows: i ranges from 1 to m, j ranges through the subsets of $[n]$ of size s, and $m_{i,j}^{(s)} = 1$ if $j \subseteq A_i$, and 0 otherwise.

Ray-Chaudhuri and Wilson proved that M_s has full row rank, i.e. $\mathrm{rk}(M_s) = m$, by showing that $M_s M_s^t$ is a non-singular matrix.

Exercise 0.22. Prove: the (i,j)-entry of $M_s M_s^t$ is

$$\binom{|A_i \cap A_j|}{s}.$$

$(1 \leq i, j \leq m.)$