Discrete Math, Sixth Problem Set (June 30)

REU 2003

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Exercise 0.1. Show if $3\alpha_1 + 7\alpha_2 - 17\alpha_3 = 0$ then $\{q\underline{\alpha} \mod 1 : q \in \mathbb{Z}\}$ lies in a finite number of hyperplanes, where $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

Exercise$^+$ 0.2. (Kronecker) If the real numbers $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then the sequence $\{q\underline{\alpha} \mod 1 : q \in \mathbb{Z}\}$ is uniformly distributed in $[0,1]^n$, where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Exercise 0.3. $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Exercise$^+$ 0.4. $1, \sqrt{2}, \ldots, \sqrt{p}$ are linearly independent over $\mathbb{Q}$.

Gram-Schmidt orthogonalization
input: $\mathbf{b}_1, \mathbf{b}_2, \ldots$ sequence of vectors in a (finite or infinite dimensional) Euclidean space.
output: $\mathbf{b}_1^*, \mathbf{b}_2^*, \ldots$
conditions:
1. $\mathbf{b}_i^* \cdot \mathbf{b}_j^* = 0$ if $i \neq j$.
2. Let $U_i = \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_i\}$. Then also $U_i = \text{span}\{\mathbf{b}_1^*, \ldots, \mathbf{b}_i^*\}$.
3. $\mathbf{b}_i - \mathbf{b}_i^* \in U_{i-1}$.

Exercise 0.5. Prove that under the stated condition, the output sequence is unique.

Exercise 0.6. Prove that Gram-Schmidt orthogonalization preserves volumes: for all k, the k-dimensional volume of the parallelepipeds spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and $\mathbf{b}_1^*, \ldots, \mathbf{b}_k^*$ are the same:

$$\text{vol}(\mathbf{b}_1, \ldots, \mathbf{b}_k) = \text{vol}(\mathbf{b}_1^*, \ldots, \mathbf{b}_k^*).$$

Definition 0.7. The **Gram matrix** of $\mathbf{b}_1, \ldots, \mathbf{b}_k$ is

$$G(\mathbf{b}_1, \ldots, \mathbf{b}_k) = (\mathbf{b}_i \cdot \mathbf{b}_j)_{k \times k}.$$

Exercise 0.8. Prove that $G(\mathbf{b}_1, \ldots, \mathbf{b}_k)$ is a positive semi-definite matrix. It is positive definite iff $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ are linearly independent.

Exercise 0.9. Show $\det G(\mathbf{b}_1, \ldots, \mathbf{b}_k) = \det G(\mathbf{b}_1^*, \ldots, \mathbf{b}_k^*)$.

Example 0.10.

$$G(\mathbf{b}_1^*, \dots, \mathbf{b}_k^*) = \begin{bmatrix} \|\mathbf{b}_1\|^2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \|\mathbf{b}_k\|^2 \end{bmatrix}$$

Exercise 0.11. Argue that $\det G(\mathbf{b}_1, \dots, \mathbf{b}_k)$ is the square of the k-dimensional volume of the paralellepiped spanned by $\mathbf{b}_1, \dots, \mathbf{b}_k$. *Hint.* Use the previous example and the exercise preceding.

Exercise 0.12. The k-dimensional volume of the parallelepiped spanned by k integral vectors in $\mathbb{R}^n$ is the square root of an integer. *Note:* If $k = n$, then the volume itself is an integer. Why?

Example 0.13. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then

$$\mathrm{vol}(\mathbf{a}, \mathbf{b})^2 = \begin{bmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{bmatrix} = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \geq 0.$$

Note that the last inequality is the Cauchy–Schwarz inequality.

Exercise 0.14. Let A be the $n \times k$ matrix of which the i-th column has the coordinates of $\mathbf{b}_i$ with respect to a fixed orthonormal basis:

$$A = [\mathbf{b}_1, \dots, \mathbf{b}_k]_{\underline{\mathbf{e}}},$$

where $\underline{\mathbf{e}}$ is an orthonormal basis. Prove: $G(\mathbf{b}_1, \dots, \mathbf{b}_k) = A^* \cdot A$. *Hint.* $A^* A = ([\mathbf{b}_i]^* [\mathbf{b}_j])$.

Exercise 0.15. Do exercises on p114–115 of handout.