Discrete Math, Eights Problem Set (July 7)

REU 2003

Instructor: Laszlo Babai
Scribe: Daniel Stefankovic

Proof of Minkowski's Theorem.

Combinatorial games. Evaluation of the game tree. Existence of a winning strategy. Proof that the first player has a winning strategy in the "Divisor" game.

Solution to the harder Erdős puzzle: If A is a set of $n+1$ numbers from 1 to $2n$ then one of them divides another.

Definition 0.1. Two elements in a partially ordered set (poset) are *comparable* if one is less than or equal to the other. A *chain* in a poset is a set of pairwise comparable elements. An *antichain* is a set of pairwise incomparable elements.

The solution to the Erdős puzzle was based on the following observation.

Exercise 0.2. The set $\{1, 2, \ldots, 2n\}$, partially ordered by divisibility, can be split into n chains.

From this it follows by the Pigeon Hole Principle that no antichain can have more than n elements, proving Erdős's claim. This is an instance of a remarkable general result at work:

Exercise$^+$ 0.3. (Dilworth's Theorem) Let P be a finite partially ordered set. Let $\alpha(P)$ be the maximum size of antichains in P, and let $\chi(P)$ denote the minimum number chains whose union is P. Then $\alpha(P) = \chi(P)$.

(Note that $\alpha(P) \leq \chi(P)$ is straightforward.)

The **comparability graph** of a poset P has P for its set of vertices; comparable elements are adjacent.

Exercise 0.4. $\alpha(P)$ is the maximum size of independent sets of the comparability graph of P; and $\chi(P)$ is the chromatic number of the complement of the comparability graph.

The next exercise shows that the $\alpha(G)$ vs. $\chi(\overline{G})$ behavior of most graphs is diametrically opposite to comparability graphs.

Exercise 0.5. Prove: there exist graphs G with n vertices and with $\alpha(G) = O(\log n)$ and $\chi(\overline{G}) = \Omega(n/\log n)$. *Hint.* Show that almost all graphs have the desired property.

Exercise 0.6. Verify that the "SETs" in the card game "SET" are lines in $\mathrm{AG}(4,3)$, the 4-dimensional affine geometry over $\mathbb{F}_3$.

Exercise 0.7. Show that there are 1080 lines in $\mathrm{AG}(4,3)$.

Definition 0.8. An *independent set* in $\mathrm{AG}(n,q)$ is a set S of points such that no line is contained in S. Let $\alpha(n,q)$ denote the maximum size of independent sets in $\mathrm{AG}(n,q)$.

Exercise 0.9. We are interested in the value of $\alpha(4,3)$, the maximum number of SET-cards without a "SET."

1. Show that $\alpha(2,3) = 4$.
2. Use this to show that $\alpha(4,3) \leq 36$.
3. Show that $\alpha(n,3) \geq 2^n$.
4. Show that $\alpha(3,3) \geq 9$.
5. Infer from the previous exercise that $\alpha(4,3) \geq 18$.
6. Show that $\alpha(4,3) \geq 20$. (This is the best lower bound known to the instructor.)
7. Show that $\alpha(3,3) = 9$.
8. Infer from the previous exercise that $\alpha(4,3) \leq 27$.
9. Prove: if S is an independent set in $\mathrm{AG}(3,3)$ and $|S| \geq 7$ then S contains 4 points which belong to a 2-dimensional affine subspace ($\mathrm{AG}(2,3)$).
10. Prove: if an independent set S in $\mathrm{AG}(4,3)$ does not contain 4 points that belong to a 2-dimensional affine subspace then $|S| \leq 15$.
11. Prove: $\alpha(4,3) \leq 24$. This is the best upper bound known to the instructor. *Hint.* Take 2-dimensional affine subspace A such that $|A \cap S| = 4$. Let B be a 3-dimensional affine subspace containing A. Then $|S \cap B \setminus A| \leq 5$. Four such sets $B_i \setminus A$ tile $\mathrm{AG}(4,3)$.
12. * Reduce the gap between the lower and upper bounds $20 \leq \alpha(4,3) \leq 24$.
13. Prove: $\alpha(n,3)\alpha(k,3) \leq \alpha(n+k,3)$.
14. Let $L = \lim_{n\to\infty} \sqrt[n]{\alpha(n,3)}$. Prove that this limit exists.

15. Prove: for all n, $L \geq \sqrt[n]{\alpha(n,3)}$.

16. Prove: $2.11 < L \leq 3$.

17. * Is $L < 3$? (The answer is not known to the instructor.)

Exercise 0.10. Let $f(x,y)$ be a two variable polynomial over $\mathbb{F}_q$ of total degree $\leq 2q-3$. If f is not identically zero then it attains non-zero values more than once.

Definition 0.11. An *blocking set* in $\mathrm{AG}(n,q)$ is a set S of points such that every line intersects S.

Note that blocking sets are the complements of the independent sets.

Exercise^{+} 0.12. Prove: $\alpha(2,q) = (q-1)^2$. (*Hint:* Suppose that there is a blocking set $\{(a_1,b_1),\ldots,(a_m,b_m)\}$ with $m \leq 2q-2$ elements. W.l.o.g. $a_1 = b_1 = 0$. Consider the polynomial $f(x,y) = (a_2x + b_2y + 1)\ldots(a_mx + b_my + 1)$.)