Discrete Math Puzzles - Second Set. July 2, 2003

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This sheet describes some exercises and projects for independent work. If you are stuck, feel free to send me email, describe your attempt at the problem, ask for a hint. Also, take advantage of the **office hours** held by the counselors TTh 7–8pm in the "Theory Lounge," Ryerson 162.

Exercise 1 (Lovász toggle). Let $G = (V, E)$ be an undirected graph. Assume every vertex of G has degree $\leq r + b + 1$. We wish to color the vertices red or blue (each vertex gets exactly one color) such that each red vertex has at most r red neighbors and each blue vertex has at most b blue neighbors. (Note that this is not a *legal* coloring in the sense of the definition of the chromatic number.) Show that this is always possible, using the following algorithm (given here in pseudocode).

procedure *Lovász-toggle*

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1 Initialize by coloring each vertex arbitrarily
2 Call a vertex "bad" if it has more than the permitted number of neighbors of its own color
3 BAD := set of bad vertices
4 while BAD ≠ ∅
5     pick a bad vertex
6     recolor it
7     update BAD
8 end(while)
```

(a) Prove that this algorithm will terminate in a finite number of steps. (Give a very simple and convincing argument, no more than 5 or 6 lines.) Give an upper bound on the number of cycles of the **while** loop in terms of the basic parameters $|V|, |E|$. *Hint.* Call the graph with a coloring a "configuration." With each configuration, associate an integer (the "potential") in such a way that each round of the Lovász-toggle reduces the potential. This will give a bound on the number of rounds. Note that "the number of bad vertices" is NOT an appropriate potential function: it can increase.

(b) Show that statement (a) becomes false if the degree bound is increased to $r + s + 2$. Construct graphs where each vertex has degree $\leq r + s + 2$ and where

- the algorithm never terminates, regardless of the initial coloring and the choice of bad vertex made in line 5;
- for some initial colorings and some choices of the bad vertex the algorithm will terminate, for others it will not.

Exercise 2 (The probability that two random integers are relatively prime is $6/\pi^2$.). Let $x, y \in \mathbb{N}$ be two positive integers picked uniformly at random. Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Assuming the limit $\Pr(\text{g.c.d.}\,(x, y) = 1) = \lim_{n\to\infty} \Pr(\text{g.c.d.}\,(x, y) = 1 \colon 1 \leq x, y \leq n)$ exists, prove that it must be $1/\zeta(2)$. (Give a three-line proof.) (see also Exercises 4.2.22, 4.2.24 from the "Basic Number Theory" handout)

Exercise 3. Let R be a rectangle partitioned into k rectangles $R_1, \ldots, R_k$, i.e. $R = R_1 \cup \cdots \cup R_k$ and the R_i are disjoint. Prove that if each of the R_i has a side of integer length then R must have a side of integer length, too.