Recall that $\chi(X)$ is the chromatic number of $X$ and $\alpha(X)$ is the independence number of $X$ (size of the largest independent set). (An independent set, or anticlique, is a set of pairwise non-adjacent vertices).

**Exercise 0.1.** Show that $\alpha(X)\chi(X)\geq n$.

**Exercise 0.2.** Show that $\chi(X)$ is not bounded above by any function of $n/\alpha(X)$.

**Exercise 0.3.** Prove: $\alpha(X) + \chi(X) \leq n + 1$.

**Exercise 0.4.** Prove: $\alpha(X)\chi(X) \leq (n + 1)^2/4$. *Hint.* Use the preceding exercise and the inequality between the geometric and arithmetic means.

**Exercise 0.5.** Prove that the bounds in the preceding two exercises are tight for all odd $n$.

This preceding exercise shows that $\chi(X)$ can be much larger (by a factor of $\Omega(n)$) than its lower bound $n/\alpha(X)$, so this lower bound is far from being tight. Contrast this with the situation for vertex-transitive graphs:

**Exercise 0.6.** If $X$ is vertex-transitive then we have nearly matching lower and upper bounds for $\chi(X)$ in terms of $n$ and $\alpha(X)$: $\chi(X) \leq \frac{n(1+\ln n)}{\alpha(X)}$.

**Definition 0.7.** A sequence $a_1, \ldots, a_n$ is **unimodal** if there is $k$ such that $a_1, \ldots, a_k$ is increasing and $a_k, \ldots, a_n$ is decreasing (not necessarily strictly). A sequence $a_1, \ldots, a_n$ is **log-concave** if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i$.

**Exercise 0.8.** If a sequence is log-concave then it is unimodal.

**Exercise 0.9.** Prove that the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.

**Definition 0.10.** A graph $X$ is **distance-transitive** if $\forall a, b, x, y \in V(X)$ if $\text{dist}(a, b) = \text{dist}(x, y)$ then $(\exists g \in \text{Aut } X)(a^g = x, b^g = y)$. 

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Exercise 0.11. Construct infinitely many connected graphs that are vertex-transitive but not distance transitive.

Exercise 0.12. Show that Petersen’s graph is distance-transitive.

Kneser’s graph $K(n, s)$, $n \geq 2s + 1$ has $\binom{n}{s}$ vertices corresponding to the subsets of $[n]$ of size $s$ with two vertices being adjacent if the corresponding sets are disjoint. Johnson’s graph $J(n, s)$, $n \geq s + 1$ has $\binom{n}{s}$ vertices corresponding to the subsets of $[n]$ of size $s$ with two vertices being adjacent if the corresponding sets have symmetric difference of size 2.

Exercise 0.13. Show that the $n$-cube, Kneser’s graph $K(n, s)$, Johnson’s graph $J(n, s)$ are distance transitive.

Let $S(x, r)$ denote the sphere of radius $r$ about vertex $x$, i.e.

$$S(x, r) = \{ y \in V(X) | \text{dist}(x, y) = r \}.$$  

Exercise 0.14. Let $X$ be distance-transitive. Let $a_r = |S(x, r)|$ for some $x \in V(X)$. (So $a_0 = 1$.) Show that the sequence $\{a_r\}$ is log-concave.

Exercise 0.15. Construct infinitely many connected vertex-transitive graphs such that the sequence sequence $\{a_r\}$ is not unimodal.

Exercise 0.16. PROJECT. How pathological can the sequence $\{a_r\}$ be for connected vertex-transitive graphs? Is it possible to have $a_1$ “large,” and $a_2$ “much larger,” $a_3$ “even larger,” then $a_4$ “much smaller” than $a_3$, and then $a_5$ much larger than $a_4$, perhaps much larger even than $a_3$? What kind of peaks and valleys can the sequence $\{a_r\}$ have? – While all these exercises are for finite graphs, can an infinite vertex-transitive graph have $a_1, a_2, a_3$ infinite, $a_4$ finite, and then $a_5$ infinite again?

Exercise 0.17. If $a_0, a_1, \ldots$ is log-concave then $(\forall i \leq j)(\forall k \geq 1)(a_{k-i}a_{j+k} \leq a_ia_j)$.

Let $B(x, r)$ be the ball of radius $r$ around vertex $x$, i.e.

$$B(x, r) = \{ y \in V(X) | \text{dist}(x, y) \leq r \}.$$  

Lemma 0.18 (Gromov). Let $X$ be a vertex-transitive graph. Let $f(r) = |B(x, r)|$ for some $x$. Then

$$f(r)f(5r) \leq f(4r)^2.$$  

In Gromov’s Lemma, $X$ may be infinite but it must be locally finite (the vertices have finite degree).

Exercise 0.19. Prove Gromov’s Lemma. (Hint: Let $Y$ be the maximal set of vertices at pairwise distance $\geq 2r + 1$ within $B(3r, x)$.

Proof: $|Y| \cdot f(r) \leq f(4r)$ and $|Y| \cdot f(4r) \geq f(5r).$)