Proof of Minkowski’s Theorem.

Combinatorial games. Evaluation of the game tree. Existence of a winning strategy. Proof that the first player has a winning strategy in the “Divisor” game.

Solution to the harder Erdős puzzle: If \( A \) is a set of \( n + 1 \) numbers from 1 to \( 2n \) then one of them divides another.

**Definition 0.1.** Two elements in a partially ordered set (poset) are *comparable* if one is less than or equal to the other. A *chain* in a poset is a set of pairwise comparable elements. An *antichain* is a set of pairwise incomparable elements.

The solution to the Erdős puzzle was based on the following observation.

**Exercise 0.2.** The set \( \{1, 2, \ldots, 2n\} \), partially ordered by divisibility, can be split into \( n \) chains.

From this it follows by the Pigeon Hole Principle that no antichain can have more than \( n \) elements, proving Erdős’s claim. This is an instance of a remarkable general result at work:

**Exercise+ 0.3. (Dilworth’s Theorem)** Let \( P \) be a finite partially ordered set. Let \( \alpha(P) \) be the maximum size of antichains in \( P \), and let \( \chi(P) \) denote the minimum number chains whose union is \( P \). Then \( \alpha(P) = \chi(P) \).

(Note that \( \alpha(P) \leq \chi(P) \) is straightforward.)

The *comparability graph* of a poset \( P \) has \( P \) for its set of vertices; comparable elements are adjacent.

**Exercise 0.4.** \( \alpha(P) \) is the maximum size of independent sets of the comparability graph of \( P \); and \( \chi(P) \) is the chromatic number of the complement of the comparability graph.
The next exercise shows that the $\alpha(G)$ vs. $\chi(G)$ behavior of most graphs is diametrically opposite to comparability graphs.

**Exercise 0.5.** Prove: there exist graphs $G$ with $n$ vertices and with $\alpha(G) = O(\log n)$ and $\chi(G) = \Omega(n/\log n)$. *Hint.* Show that almost all graphs have the desired property.

**Exercise 0.6.** Verify that the “SETs” in the card game “SET” are lines in $AG(4,3)$, the 4-dimensional affine geometry over $F_3$.

**Exercise 0.7.** Show that there are 1080 lines in $AG(4,3)$.

**Definition 0.8.** An *independent set* in $AG(n,q)$ is a set $S$ of points such that no line is contained in $S$. Let $\alpha(n,q)$ denote the maximum size of independent sets in $AG(n,q)$.

**Exercise 0.9.** We are interested in the value of $\alpha(4,3)$, the maximum number of SET-cards without a “SET.”

1. Show that $\alpha(2,3) = 4$.
2. Use this to show that $\alpha(4,3) \leq 36$.
3. Show that $\alpha(n,3) \geq 2^n$.
4. Show that $\alpha(3,3) \geq 9$.
5. Infer from the previous exercise that $\alpha(4,3) \geq 18$.
6. Show that $\alpha(4,3) \geq 20$. (This is the best lower bound known to the instructor.)
7. Show that $\alpha(3,3) = 9$.
8. Infer from the previous exercise that $\alpha(4,3) \leq 27$.
9. Prove: if $S$ is an independent set in $AG(3,3)$ and $|S| \geq 7$ then $S$ contains 4 points which belong to a 2-dimensional affine subspace ($AG(2,3)$).
10. Prove: if an independent set $S$ in $AG(4,3)$ does not contain 4 points that belong to a 2-dimensional affine subspace then $|S| \leq 15$.
11. Prove: $\alpha(4,3) \leq 24$. This is the best upper bound known to the instructor. *Hint.* Take 2-dimensional affine subspace $A$ such that $|A \cap S| = 4$. Let $B$ be a 3-dimensional affine subspace containing $A$. Then $|S \cap B \setminus A| \leq 5$. Four such sets $B_i \setminus A$ tile $AG(4,3)$.
12. * Reduce the gap between the lower and upper bounds $20 \leq \alpha(4,3) \leq 24$.
13. Prove: $\alpha(n,3)\alpha(k,3) \leq \alpha(n + k,3)$.
14. Let $L = \lim_{n \to \infty} \sqrt[n]{\alpha(n,3)}$. Prove that this limit exists.
15. Prove: for all \( n \), \( L \geq \sqrt[3]{\alpha(n,3)} \).

16. Prove: \( 2.11 < L \leq 3 \).

17. * Is \( L < 3 \) ? (The answer is not known to the instructor.)

**Exercise 0.10.** Let \( f(x, y) \) be a two variable polynomial over \( F_q \) of total degree \( \leq 2q - 3 \). If \( f \) is not identically zero then it attains non-zero values more than once.

**Definition 0.11.** An *blocking set* in \( AG(n, q) \) is a set \( S \) of points such that every line intersects \( S \).

Note that blocking sets are the complements of the independent sets.

**Exercise 0.12**. Prove: \( \alpha(2, q) = (q - 1)^2 \). *(Hint: Suppose that there is a blocking set \( \{(a_1, b_1), \ldots, (a_m, b_m)\} \) with \( m \leq 2q - 2 \) elements. W.l.o.g. \( a_1 = b_1 = 0 \). Consider the polynomial \( f(x, y) = (a_2 x + b_2 y + 1) \ldots (a_m x + b_m y + 1) \).)*