Linear Algebra, 1st day, Monday 6/28/04
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1 Vector spaces, linear independence

Definition 1.1. A **vector space** is a set V with

(a) addition $V \times V \to V$, $(x, y) \mapsto x + y$, and

(b) scalar multiplication $\mathbb{R} \times V \to V$, $(\alpha, x) \mapsto \alpha x$,

satisfying following axioms

(a) $(V, +)$ is an abelian group, i. e.

- (a1) $(\forall x, y \in V)(\exists!\, x + y \in V)$,
- (a2) $(\forall x, y \in V)(x + y = y + x)$ (commutative law),
- (a3) $(\forall x, y, z \in V)((x + y) + z = x + (y + z))$ (associative law),
- (a4) $(\exists 0 \in V)(\forall x)(x + 0 = 0 + x = x)$ (existence of zero),
- (a5) $(\forall x \in V)(\exists(-x) \in V)(x + (-x) = 0)$,

b) (b1) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)(\alpha(\beta x)) = (\alpha\beta)x$ ("associativity" linking two operations),
- (b2) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)((\alpha + \beta)x = \alpha x + \beta x)$ (distributivity over scalar addition),
- (b3) $(\forall \alpha \in \mathbb{R})(\forall x, y \in V)(\alpha(x + y) = \alpha x + \alpha y)$ (distributivity over vector addition),

(c) $(\forall x \in V)(1 \cdot x = x)$ (normalization).

Exercise 1.2. Show $(\forall x \in V)(0x = 0)$. (The first 0 is a number, the second a vector.)

Exercise 1.3. Show $(\forall \alpha \in \mathbb{R})(\alpha 0 = 0)$.

Exercise 1.4. Show $(\forall \alpha \in \mathbb{R})(\forall x \in V)(\alpha x = 0 \Leftrightarrow (\alpha = 0 \text{ or } x = 0))$

Definition 1.5. A **linear combination** of vectors $v_1, \ldots, v_k \in V$ is a vector $\alpha_1 v_1 + \cdots + \alpha_k v_k$ where $\alpha_1, \ldots \alpha_k \in \mathbb{R}$. The **span** of $v_1, \ldots, v_k \in V$ is the set of all linear combinations of $v_1, \ldots, v_k$, i. e., $\text{Span}(v_1, \ldots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$.

Remark 1.6. We let $\text{Span}(\emptyset) = \{0\}$.

Remark 1.7. A linear combination of an infinite set of vectors $S \subseteq V$ is a linear combination of a finite subset of S.

Note that 0 is always in $\text{Span}(v_1, \ldots, v_k)$ because the trivial linear combination $(\forall i)\alpha_i = 0$ is $0 \cdot v_1 + \cdots + 0 \cdot v_k = 0$.

Definition 1.8. Vectors $v_1, \ldots, v_k \in V$ are **linearly independent** if only the trivial linear combination gives 0, i. e., $\alpha_1 v_1 + \ldots \alpha_k v_k = 0 \Rightarrow \alpha_1 = \cdots = \alpha_k = 0$.

Exercise 1.9. Which one element sets of vectors are linearly independent?

Exercise 1.10. Show that if $T \subseteq S \subseteq V$ and S is linearly independent then T is linearly independent.

We say that vectors $u, v \in V$ are **parallel** if $u, v \neq 0$ and $\exists \alpha \in \mathbb{R}$ such that $u = \alpha v$.

Exercise 1.11. Show that vectors $u, v \in V$ are linearly dependent if and only if $a = 0$ or $b = 0$ or a, b are parallel.

Exercise 1.12. An infinite set of vectors is linearly independent if and only if all finite subsets are linearly independent.

Remark 1.13. We say that a property P is a **finitary property** if a set S has the property P if and only if all finite subsets of S have property P.

Exercise 1.14. * **(Erdős – deBruijn)** Show that 3-colorability of a graph is a finitary property. (The same holds for 4-colorability, etc.)

The set of all polynomials with real coefficients is a vector space $\mathbb{R}[x]$.

Exercise 1.15. Show that $1, x, x^2, \ldots$ are linearly independent.

Definition 1.16. The polynomial $f(x) = \sum a_i x^i$ has **degree** k if $a_k \neq 0$, but $(\forall j > k)(a_j = 0)$. Notation: $\deg(f) = k$. We let $\deg(0) = -\infty$. Note: the nonzero constant polynomials have degree 0.

Exercise 1.17. Prove: $\deg(fg) = \deg(f) + \deg(g)$. (Note that this remians true if one of the polynomials f, g is the zero polynomial.)

Exercise 1.18. Prove: $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

Exercise 1.19. Prove that if $f_0, f_1, f_2, \dots$ is a sequence of polynomials, $\deg(f_i) = i$ then $f_0, f_1, f_2, \dots$ are linearly independent.

Exercise 1.20. Let $f(x) = (x - \alpha_1)...(x - \alpha_k)$ where $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $g_i(x) = f(x)/(x - \alpha_i)$. Show that $g_1, \dots, g_k$ are linearly independent.

Exercise 1.21. Prove: for all $\alpha, \beta \in \mathbb{R}$, $\sin(x), \sin(x + \alpha), \sin(x + \beta)$ are linearly dependent functions $\mathbb{R} \to \mathbb{R}$.

Exercise 1.22. Prove: $1, \sin(x), \sin(2x), \sin(3x), \dots, \cos(x), \cos(2x), \dots$ are linearly independent functions $\mathbb{R} \to \mathbb{R}$.

Definition 1.23. A **maximal** linearly independent subset of a set $S \subseteq V$ is a subset $T \subseteq S$ such that

(a) T is linearly independent, and

(b) if $T \subsetneq T' \subseteq S$ then T' is linearly dependent.

Definition 1.24. A **maximum** linearly independent subset of a set $S \subseteq V$ is a subset $T \subseteq S$ such that

(a) T is linearly independent, and

(b) if $T' \subseteq S$ is linearly independent then $|T| \geq |T'|$.

Exercise 1.25. (Independence of vertices in a graph.) Show that 6-cycle, there exists a maximum independent set of vertices which is not maximal.

We shall see that this cannot happen with linear independence: every maximal linearly independent set is maximum.

Exercise 1.26. Let $S \subseteq V$. Then there exists $T \subseteq S$ such that T is a maximal independent subset of S.

Exercise 1.27. Let $L \subseteq S \subseteq V$. Assume L is linearly independent. Then there exists a maximal linearly independent subset $T \subseteq S$ such that $L \subseteq T$. (Every linearly independent subset of S set can be extended to a maximal linearly independent subset of S.)

Remark 1.28. This is easy to prove Ex. **??** by successively adding vectors until our set becomes maximal as long as all linearly independent subsets of S are finite. For the infinite case, we need an axiom from set theory called Zorn's Lemma (a version of the Axiom of Choice).

Definition 1.29. A vector $v \in V$ **depends** on $S \subseteq V$ if $v \in \text{Span}(S)$, i.e. v is a linear combination of S.

Definition 1.30. A set of vectors $T \subseteq V$ **depends** on $S \subseteq V$ if $T \subseteq \text{Span}(S)$.

Exercise 1.31. Show that dependence is transitive: if $R \subseteq \text{Span}(T)$ and $T \subseteq \text{Span}(S)$ then $R \subseteq \text{Span}(S)$.

Exercise 1.32. Suppose that $\sum \alpha_i v_i$ is a nontrivial linear combination. Then $(\exists i)$ such that v_i depends on the rest (i. e. on $\{v_j \mid j \neq i\}$). Indeed, this will be the case whenever $\alpha_i \neq 0$.

Exercise 1.33. If $v_1, \dots, v_k$ are linearly independent and $v_1, \dots, v_k, v_{k+1}$ are linearly dependent then v_{k+1} depends on $v_1, \dots, v_k$.

Theorem 1.34 (Fundamental Fact of Linear Algebra). *If $v_1, \dots, v_k$ are linearly independent and $v_1, \dots, v_k \in \text{Span}(w_1, \dots, w_\ell)$ then $k \leq \ell$.*

Corollary 1.35. *All maximal independent sets are maximum.*

Exercise 1.36. If $T \subseteq S$, T is a maximal independent subset of S then $S \subseteq \text{Span}(T)$.

Exercise 1.37. Prove Corollary **??** from Theorem **??** and Exercise **??**.

Definition 1.38. For $S \subseteq V$, the **rank** of S is the common cardinality of all the maximal independent subsets of S. Notation: $\text{rk}(S)$.

Definition 1.39. The **dimension** of a vector space is $\dim(V) := \text{rk}(V)$.

Exercise 1.40. Show that $\dim(\mathbb{R}^n) = n$.

Exercise 1.41. Let P_k be the space of polynomials of degree $\leq k$. Show that $\dim(P_k) = k+1$.

Exercise 1.42. Let $T = \{\sin(x + \alpha) \mid \alpha \in \mathbb{R}\}$. Prove $\text{rk}(T) = 2$.

2 Basis

Definition 1.43. A **basis** of V is a linearly independent set which spans V.

Definition 1.44. A **basis** of $S \subseteq V$ is a linearly independent subset of S which spans S. In other words, a basis B of S is a linearly independent set satisfying $B \subseteq S \subseteq \text{Span}(B)$.

Exercise 1.45. B is a basis of S if and only if B is a maximal independent subset of S.

Exercise 1.46. Prove: if B is a basis of V then $\dim(V) = |B|$.

Exercise 1.47. A "Fibonacci-type sequence" is a sequence $(a_0, a_1, a_2, \dots)$ such that $(\forall n)(a_{n+2} = a_{n+1} + a_n)$.

(a) Prove that the Fibonacci-type sequences form a 2-dimensional vector space.

(b) Find a basis in this space consisting of two geometric progressions.

(c) Express the Fibonacci sequence $(0, 1, 1, 2, 3, 5, 8, 13, \dots)$ as a linear combination of the basis found in item (b).