1 Vector spaces, linear independence

Definition 1.1. A vector space is a set $V$ with

(a) addition $V \times V \to V$, $(x, y) \mapsto x + y$, and

(b) scalar multiplication $\mathbb{R} \times V \to V$, $(\alpha, x) \mapsto \alpha x$,

satisfying following axioms

(a) $(V, +)$ is an abelian group, i.e.

(a1) $(\forall x, y \in V)(\exists ! x + y \in V)$,
(a2) $(\forall x, y \in V)(x + y = y + x)$ (commutative law),
(a3) $(\forall x, y, z \in V)((x + y) + z = x + (y + z))$ (associative law),
(a4) $(\exists 0 \in V)(\forall x)(x + 0 = 0 + x = x)$ (existence of zero),
(a5) $(\forall x \in V)(\exists (-x) \in V)(x + (-x) = 0)$,

b) (b1) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)(\alpha(\beta x)) = (\alpha \beta) x$ ("associativity" linking two operations),
(b2) $(\forall \alpha, \beta \in \mathbb{R})(\forall x \in V)((\alpha + \beta)x = \alpha x + \beta x)$ (distributivity over scalar addition),
(b3) $(\forall \alpha \in \mathbb{R})(\forall x, y \in V)(\alpha(x + y) = \alpha x + \alpha y)$ (distributivity over vector addition),

(c) $(\forall x \in V)(1 \cdot x = x)$ (normalization).

Exercise 1.2. Show $(\forall x \in V)(0x = 0)$. (The first 0 is a number, the second a vector.)

Exercise 1.3. Show $(\forall \alpha \in \mathbb{R})(\alpha 0 = 0)$.

Exercise 1.4. Show $(\forall \alpha \in \mathbb{R})(\forall x \in V)(\alpha x = 0 \leftrightarrow (\alpha = 0 \text{ or } x = 0))$
Definition 1.5. A **linear combination** of vectors \( v_1, \ldots, v_k \in V \) is a vector \( \alpha_1 v_1 + \cdots + \alpha_k v_k \) where \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \). The **span** of \( v_1, \ldots, v_k \in V \) is the set of all linear combinations of \( v_1, \ldots, v_k \), i.e., \( \text{Span}(v_1, \ldots, v_k) = \{ \alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R} \} \).

**Remark 1.6.** We let \( \text{Span}(\emptyset) = \{0\} \).

**Remark 1.7.** A linear combination of an infinite set of vectors \( S \subseteq V \) is a linear combination of a finite subset of \( S \).

Note that 0 is always in \( \text{Span}(v_1, \ldots, v_k) \) because the trivial linear combination \( (\forall i) \alpha_i = 0 \) is \( 0 \cdot v_1 + \cdots + 0 \cdot v_k = 0 \).

**Definition 1.8.** Vectors \( v_1, \ldots, v_k \in V \) are **linearly independent** if only the trivial linear combination gives 0, i.e., \( \alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \cdots = \alpha_k = 0 \).

**Exercise 1.9.** Which one element sets of vectors are linearly independent?

**Exercise 1.10.** Show that if \( T \subseteq S \subseteq V \) and \( S \) is linearly independent then \( T \) is linearly independent.

We say that vectors \( u, v \in V \) are **parallel** if \( u, v \neq 0 \) and \( \exists \alpha \in \mathbb{R} \) such that \( u = \alpha v \).

**Exercise 1.11.** Show that vectors \( u, v \in V \) are linearly dependent if and only if \( a = 0 \) or \( b = 0 \) or \( a, b \) are parallel.

**Exercise 1.12.** An infinite set of vectors is linearly independent if and only if all finite subsets are linearly independent.

**Remark 1.13.** We say that a property \( P \) is a **finitary property** if a set \( S \) has the property \( P \) if and only if all finite subsets of \( S \) have property \( P \).

**Exercise 1.14.** *(Erdős – deBruijn)* Show that 3-colorability of a graph is a finitary property. (The same holds for 4-colorability, etc.)

The set of all polynomials with real coefficients is a vector space \( \mathbb{R}[x] \).

**Exercise 1.15.** Show that \( 1, x, x^2, \ldots \) are linearly independent.

**Definition 1.16.** The polynomial \( f(x) = \sum a_k x^k \) has **degree** \( k \) if \( a_k \neq 0 \), but \( (\forall j > k)(a_j = 0) \). Notation: \( \deg(f) = k \). We let \( \deg(0) = -\infty \). Note: the nonzero constant polynomials have degree 0.

**Exercise 1.17.** Prove: \( \deg(fg) = \deg(f) + \deg(g) \). (Note that this remains true if one of the polynomials \( f, g \) is the zero polynomial.)

**Exercise 1.18.** Prove: \( \deg(f + g) \leq \max\{\deg(f), \deg(g)\} \).
Exercise 1.19. Prove that if $f_0, f_1, f_2, \ldots$ is a sequence of polynomials, $\deg(f_i) = i$ then $f_0, f_1, f_2, \ldots$ are linearly independent.

Exercise 1.20. Let $f(x) = (x - \alpha_1) \cdots (x - \alpha_k)$ where $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $g_k(x) = f(x)/(x - \alpha_k)$. Show that $g_1, \ldots, g_k$ are linearly independent.

Exercise 1.21. Prove: for all $\alpha, \beta \in \mathbb{R}$, $\sin(x), \sin(x+\alpha), \sin(x+\beta)$ are linearly dependent functions $\mathbb{R} \to \mathbb{R}$.

Exercise 1.22. Prove: $1, \sin(x), \sin(2x), \sin(3x), \ldots, \cos(x), \cos(2x), \ldots$ are linearly independent functions $\mathbb{R} \to \mathbb{R}$.

Definition 1.23. A **maximal** linearly independent subset of a set $S \subseteq V$ is a subset $T \subseteq S$ such that

(a) $T$ is linearly independent, and

(b) if $T \subseteq T' \subseteq S$ then $T'$ is linearly dependent.

Definition 1.24. A **maximum** linearly independent subset of a set $S \subseteq V$ is a subset $T \subseteq S$ such that

(a) $T$ is linearly independent, and

(b) if $T' \subseteq S$ is linearly independent then $|T| \geq |T'|$.

Exercise 1.25. (Independence of vertices in a graph.) Show that 6-cycle, there exists a maximum independent set of vertices which is not maximal.

We shall see that this cannot happen with linear independence: every maximal linearly independent set is maximum.

Exercise 1.26. Let $S \subseteq V$. Then there exists $T \subseteq S$ such that $T$ is a maximal independent subset of $S$.

Exercise 1.27. Let $L \subseteq S \subseteq V$. Assume $L$ is linearly independent. Then there exists a maximal linearly independent subset $T \subseteq S$ such that $L \subseteq T$. (Every linearly independent subset of $S$ set can be extended to a maximal linearly independent subset of $S$.)

Remark 1.28. This is easy to prove Ex. ?? by successively adding vectors until our set becomes maximal as long as all linearly independent subsets of $S$ are finite. For the infinite case, we need an axiom from set theory called Zorn’s Lemma (a version of the Axiom of Choice).

Definition 1.29. A vector $v \in V$ **depends** on $S \subseteq V$ if $v \in \text{Span}(S)$, i.e. $v$ is a linear combination of $S$.

Definition 1.30. A set of vectors $T \subseteq V$ **depends** on $S \subseteq V$ if $T \subseteq \text{Span}(S)$.
Exercise 1.31. Show that dependence is transitive: if \( R \subseteq \text{Span}(T) \) and \( T \subseteq \text{Span}(S) \) then \( R \subseteq \text{Span}(S) \).

Exercise 1.32. Suppose that \( \sum \alpha_i v_i \) is a nontrivial linear combination. Then (\( \exists i \)) such that \( v_i \) depends on the rest (i.e. on \( \{v_j \mid j \neq i\} \)). Indeed, this will be the case whenever \( \alpha_i \neq 0 \).

Exercise 1.33. If \( v_1, \ldots, v_k \) are linearly independent and \( v_1, \ldots, v_k, v_{k+1} \) are linearly dependent then \( v_{k+1} \) depends on \( v_1, \ldots, v_k \).

Theorem 1.34 (Fundamental Fact of Linear Algebra). If \( v_1, \ldots, v_k \) are linearly independent and \( v_1, \ldots, v_k \in \text{Span}(w_1, \ldots, w_\ell) \) then \( k \leq \ell \).

Corollary 1.35. All maximal independent sets are maximum.

Exercise 1.36. If \( T \subseteq S, T \) is a maximal independent subset of \( S \) then \( S \subseteq \text{Span}(T) \).

Exercise 1.37. Prove Corollary ?? from Theorem ?? and Exercise ??.

Definition 1.38. For \( S \subseteq V \), the rank of \( S \) is the common cardinality of all the maximal independent subsets of \( S \). Notation: \( \text{rk}(S) \).

Definition 1.39. The dimension of a vector space is \( \dim(V) := \text{rk}(V) \).

Exercise 1.40. Show that \( \dim(\mathbb{R}^n) = n \).

Exercise 1.41. Let \( P_k \) be the space of polynomials of degree \( \leq k \). Show that \( \dim(P_k) = k + 1 \).

Exercise 1.42. Let \( T = \{\sin(x + \alpha) \mid \alpha \in \mathbb{R}\} \). Prove \( \text{rk}(T) = 2 \).

2 Basis

Definition 1.43. A basis of \( V \) is a linearly independent set which spans \( V \).

Definition 1.44. A basis of \( S \subseteq V \) is a linearly independent subset of \( S \) which spans \( S \). In other words, a basis \( B \) of \( S \) is a linearly independent set satisfying \( B \subseteq S \subseteq \text{Span}(B) \).

Exercise 1.45. \( B \) is a basis of \( S \) if and only if \( B \) is a maximal independent subset of \( S \).

Exercise 1.46. Prove: if \( B \) is a basis of \( V \) then \( \dim(V) = |B| \).

Exercise 1.47. A “Fibonacci-type sequence” is a sequence \( (a_0, a_1, a_2, \ldots) \) such that \( (\forall n)(a_{n+2} = a_{n+1} + a_n) \).

(a) Prove that the Fibonacci-type sequences form a 2-dimensional vector space.

(b) Find a basis in this space consisting of two geometric progressions.

(c) Express the Fibonacci sequence \( (0, 1, 1, 2, 3, 5, 8, 13, \ldots) \) as a linear combination of the basis found in item (b).