Linear Algebra, 2nd day, Tuesday 6/29/04
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1 Subspaces

Definition 2.1. A subset $U \subseteq V$ is a **subspace** (written $U \leq V$) if

(a) $0 \in U$

(b) $(\forall \alpha \in \mathbb{R})(\alpha u \in U)$

(c) $(\forall u, v \in U)(u + v \in U)$

Exercise 2.2. Show $\forall S \subseteq V$, $\text{Span}(S) \leq V$, i.e. that $\text{Span}(S)$ is a subspace of V. (Recall that $\text{Span}(\emptyset) = \{0\}$.)

Exercise 2.3. Show that $\text{Span}(S)$ is the smallest subspace containing S, i.e.

(a) $\text{Span}(S) \leq V$ and $\text{Span}(S) \supseteq S$.

(b) If $W \leq V$ and $S \subseteq W$ then $\text{Span}(S) \leq W$.

Corollary 2.4. $\text{Span}(S) = \bigcap_{S \subseteq W \leq V} W$.

Exercise 2.5. Show that the intersection of any set of subspaces is a subspace.

Remark 2.6. Note that this doesn't hold true for unions.

Exercise 2.7. If $U_1, U_2 \leq V$ then $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

2 All bases are equal

Theorem 2.8 (Fundamental Fact of Linear Algebra). *If $L, M \subseteq V$ with L is a linearly independent set of vectors, and $L \leq \text{Span}(M)$ then $|L| \leq |M|$.*

Lemma 2.9 (Steinitz Exchange Principle). *If $v_1, \ldots, v_k$ are linearly independent and $v_1, \ldots, v_k \in \text{Span}(w_1, \ldots, w_\ell)$ then $\exists j, 1 \leq j \leq \ell$ such that $w_j, v_2, \ldots, v_k$ are linearly independent. (Note in particular that $w_j \neq v_2, \ldots, v_k$.)*

Exercise 2.10. Prove the Steinintz Exchange Principle.

Exercise 2.11. Prove the Fundamental Fact using the Steinitz Echange Principle.

Definition 2.12. An $m \times n$ **matrix** is an $m \times n$ array of numbers $\{\alpha_{ij}\}$ which we write as

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

Definition 2.13. The **row (respectively column) rank** of a matrix is the rank of the set of row (respectively column) vectors.

Theorem 2.14 (The Most Amazing Fact of Basic Linear Algebra). *The row rank of a matrix is equal to its column rank. (To be proven later in today.)*

Definition 2.15. Let $S \subseteq V$, then a subset $B \subseteq S$ is a **basis** of S if

(a) B is linearly independent.

(b) $S \leq \text{Span}(B)$.

Exercise 2.16. Show that the statement: **"All bases for S have equal size"** is equivalent to Theorem **??**.

Definition 2.17. We call the common size of all bases of S the **rank** of S, denoted $\text{rk}(S)$.

3 Coordinates

Exercise 2.18. Show that $B \subseteq S$ is a basis if and only if B is a maximal linearly independent subset of S.

Exercise 2.19. If B is a basis of S then $\forall x \in S$ there exists a unique linear combination of elements in B that sums to x. In other words for all x there are unique scalars β_i such that

$$x = \sum_{i=1}^{k} \beta_i b_i.$$

Definition 2.20. For a basis B, regarded as an ordered set of vectors, we associate to each $x \in S$ the column vector

$$[x]_B := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

called the **coordinates** of x, where the β_i are as above.

4 Linear maps, isomorphism of vector spaces

Definition 2.21. Let V and W be vector spaces. We say that a map $f : V \to W$ is a **homomorphism** or a **linear map** if

(a) $(\forall x, y \in V)(f(x+y) = f(x) + f(y))$

(b) $(\forall x \in V)(\forall \alpha \in \mathbb{R})(f(\alpha x) = \alpha f(x))$

Exercise 2.22. Show that if f is a linear map then $f(0) = 0$.

Exercise 2.23. Show that $f(\sum_{i=1}^{k} \alpha_i v_i) = \sum_{i=1}^{k} \alpha_i f(v_i)$.

Definition 2.24. We say that f is an **isomorphism** if f is a bijective homomorphism.

Definition 2.25. Two spaces V and W are **isomorphic** if there exists an isomorphism between them.

Exercise 2.26. Show the relation of being isomorphic is an equivalence relation.

Exercise 2.27. Show that an isomorphism maps bases to bases.

Theorem 2.28. *If* $\dim(V) = n$ *then* $V \cong \mathbb{R}^n$.

Proof: Choose a basis, B of V, now map each vector to its coordinate vector, i.e. $v \mapsto [v]_B$.

Definition 2.29. We denote the **image** of f as the set

$$\operatorname{im}(f) = \{f(x) : x \in V\}$$

Definition 2.30. We denote the **kernel** of f as the set

$$\ker(f) = \{x \in V : f(x) = 0\}$$

Exercise 2.31. For a linear map $f : V \to W$ show that $\operatorname{im}(f) \leq W$ and $\ker(f) \leq V$.

Theorem 2.32. *For a linear map* $f : V \to W$ *we have*

$$\dim \ker(f) + \dim \operatorname{im}(f) = \dim V.$$

Lemma 2.33. *If* $U \leq V$ *and* A *is a basis of* U *then* A *can be extended to a basis of* V.

Exercise 2.34. Prove Theorem **??**. HINT: apply Lemma **??** setting $U = \ker(f)$.

5 Vector spaces over number fields

Definition 2.35. A subset $\mathbb{F} \subseteq \mathbb{C}$ is a **number field** if $\mathbb{F}$ is closed under the four arithmetic operations, i.e. for $\alpha, \beta \in \mathbb{F}$

(a) $\alpha \pm \beta \in \mathbb{F}$

(b) $\alpha\beta \in \mathbb{F}$

(c) $\frac{\alpha}{\beta} \in \mathbb{F}$ (assuming $\beta \neq 0$).

Exercise 2.36. Show that if $\mathbb{F}$ is a number field then $Q \subseteq \mathbb{F}$.

Exercise 2.37. Show that $\mathbb{Q}[\sqrt{2}]$ is a number field.

Exercise 2.38. Show that $Q[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is a number field.

Exercise 2.39 (Vector Spaces over Number Fields). Convince yourself that all of the things we have said about vector spaces remain valid if we replace $\mathbb{R}$ and $\mathbb{F}$.

Exercise 2.40. Show that if $\mathbb{F}, G$ are number fields and $\mathbb{F} \subseteq G$ then G is a vector space over $\mathbb{F}$.

Exercise 2.41. Show that $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Exercise 2.42. Show that $\dim_{\mathbb{Q}} \mathbb{R}$ has the cardinality of "continuum," that is, it has the same cardinality as $\mathbb{R}$.

Exercise 2.43 (Cauchy's Equation). We consider functions $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy's Equation: $f(x + y) = f(x) + f(y)$ with $x, y \in \mathbb{R}$. For such a function prove that

(a) If f is continuous then $f(x) = cx$.

(b) If f is continuous at a point then $f(x) = cx$.

(c) If f is bounded on some interval then $f(x) = cx$.

(d) If f is measurable in some interval then $f(x) = cx$.

(e) There exists a $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) \neq cx$ but $g(x + y) = g(x) + g(y)$. (Hint: Use the fact that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. Use a basis of this vector space. Such a basis is called a **Hamel basis**.

Exercise 2.44. Show that $1, \sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Exercise 2.45. Show that $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}$ and $\sqrt{30}$ are linearly independent over $\mathbb{Q}$.

Exercise 2.46. * Show that the set of square roots of all of the square-free integers are linearly independent over $\mathbb{Q}$. (An integer is **square free** if it is not divisible by the square of any prime number. For instance, 30 is square free but 18 is not.)

Definition 2.47. A **rational function** f over $\mathbb{R}$ is a fraction of the form

$$f(x) = \frac{g(x)}{h(x)}$$

where $g, h \in \mathbb{R}[x]$ (that is g, h are real polynomials) and $h(x) \neq 0$ (h is not the identically zero polynomial). More precisely, a rational function is an equivalence class of fractions of polynomials, where the fractions $\frac{g_1(x)}{h_1(x)}$ and $\frac{g_2(x)}{h_2(x)}$ are equivalent if and only if $g_1 \cdot h_2 = g_2 \cdot h_1$. (This is analogous to the way fractions of integers represent rational numbers; the fractions $3/2$ and $6/4$ represent the same rational number.) We denote the set of all rational functions as $\mathbb{R}(x)$.

Note that a rational function is not a function; it is an equivalence class of formal quotients.

Exercise 2.48. Prove that the rational functions $\{\frac{1}{x-\alpha} : \alpha \in \mathbb{R}\}$ are linearly independent set over $\mathbb{R}(x)$.

Corollary 2.49. $\dim_{\mathbb{R}[x]} \mathbb{R}(x)$ *has the cardinality of "conyinuum" (the same cardinality as* $\mathbb{R}$*).*

6 Elementary operations

Definition 2.50. The following actions on a set of vectors $\{v_1, \ldots, v_k\}$ are called **elementary operations:**

(a) Replace v_i by $v_i - \alpha v_j$ where $i \neq j$.

(b) Replace v_i by αv_i where $\alpha \neq 0$.

(c) Switch v_i and v_j.

Exercise 2.51. Show that the rank of a list of vectors doesn't change under elementary operations.

Exercise 2.52. Let $\{v_1, \ldots, v_k\}$ have rank r. Show that by a sequence of elementary operations we can get from $\{v_1, \ldots, v_k\}$ to a set $\{w_1, \ldots, w_k\}$ such that $w_1, \ldots, w_r$ are linearly independent and $w_{r+1} = \cdots = w_k = 0$.

Consider a matrix. An **elementary row-operation** is an elementary operation applied to the rows of the matrix. Elementary column operations are defined analogously. Exercise **??** shows that **elementary row-operations** do not change the **row-rank** of A.

Exercise 2.53. Show that elementary **row-operations** do not change the **column-rank** of a matrix.

Exercise 2.54. Use Exercises **??** and **??** prove the "amazing" Theorem **??**.