1 Subspaces

**Definition 2.1.** A subset $U \subseteq V$ is a **subspace** (written $U \leq V$) if

(a) $0 \in U$

(b) $(\forall \alpha \in \mathbb{R})(\alpha u \in U)$

(c) $(\forall u, v \in U)(u + v \in U)$

**Exercise 2.2.** Show $\forall S \subseteq V$, $\text{Span}(S) \leq V$, i.e. that $\text{Span}(S)$ is a subspace of $V$. (Recall that $\text{Span}(\emptyset) = \{0\}$.)

**Exercise 2.3.** Show that $\text{Span}(S)$ is the smallest subspace containing $S$, i.e.

(a) $\text{Span}(S) \leq V$ and $\text{Span}(S) \supseteq S$.

(b) If $W \leq V$ and $S \subseteq W$ then $\text{Span}(S) \leq W$.

**Corollary 2.4.** $\text{Span}(S) = \bigcap_{S \subseteq W \leq V} W$.

**Exercise 2.5.** Show that the intersection of any set of subspaces is a subspace.

**Remark 2.6.** Note that this doesn’t hold true for unions.

**Exercise 2.7.** If $U_1, U_2 \leq V$ then $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. 
2 All bases are equal

**Theorem 2.8** (Fundamental Fact of Linear Algebra). If $L, M \subseteq V$ with $L$ is a linearly independent set of vectors, and $L \leq \text{Span}(M)$ then $|L| \leq |M|$.

**Lemma 2.9** (Steinitz Exchange Principle). If $v_1, \ldots, v_k$ are linearly independent and $v_1, \ldots, v_k \in \text{Span}(w_1, \ldots, w_\ell)$ then $\exists j, 1 \leq j \leq \ell$ such that $w_j, v_2, \ldots, v_k$ are linearly independent. (Note in particular that $w_j \neq v_2, \ldots, v_k$.)

**Exercise 2.10.** Prove the Steinitz Exchange Principle.

**Exercise 2.11.** Prove the Fundamental Fact using the Steinitz Exchange Principle.

**Definition 2.12.** An $m \times n$ matrix is an $m \times n$ array of numbers $\{\alpha_{ij}\}$ which we write as

$$
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}
$$

**Definition 2.13.** The row (respectively column) rank of a matrix is the rank of the set of row (respectively column) vectors.

**Theorem 2.14** (The Most Amazing Fact of Basic Linear Algebra). The row rank of a matrix is equal to its column rank. (To be proven later in today.)

**Definition 2.15.** Let $S \subseteq V$, then a subset $B \subseteq S$ is a basis of $S$ if

(a) $B$ is linearly independent.

(b) $S \leq \text{Span}(B)$.

**Exercise 2.16.** Show that the statement: “All bases for $S$ have equal size” is equivalent to Theorem ??.

**Definition 2.17.** We call the common size of all bases of $S$ the rank of $S$, denoted $\text{rk}(S)$.

3 Coordinates

**Exercise 2.18.** Show that $B \subseteq S$ is a basis if and only if $B$ is a maximal linearly independent subset of $S$.

**Exercise 2.19.** If $B$ is a basis of $S$ then $\forall x \in S$ there exists a unique linear combination of elements in $B$ that sums to $x$. In other words for all $x$ there are unique scalars $\beta_i$ such that

$$
x = \sum_{i=1}^{k} \beta_i b_i.
$$
Definition 2.20. For a basis $B$, regarded as an ordered set of vectors, we associate to each $x \in S$ the column vector

$$[x]_B := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$$

called the coordinates of $x$, where the $\beta_i$ are as above.

4 Linear maps, isomorphism of vector spaces

Definition 2.21. Let $V$ and $W$ be vector spaces. We say that a map $f : V \to W$ is a homomorphism or a linear map if

(a) $(\forall x, y \in V)(f(x + y) = f(x) + f(y))$
(b) $(\forall x \in V)(\forall \alpha \in \mathbb{R})(f(\alpha x) = \alpha f(x))$

Exercise 2.22. Show that if $f$ is a linear map then $f(0) = 0$.

Exercise 2.23. Show that $f(\sum_{i=1}^{k} \alpha_i v_i) = \sum_{i=1}^{k} \alpha_i f(v_i)$.

Definition 2.24. We say that $f$ is an isomorphism if $f$ is a bijective homomorphism.

Definition 2.25. Two spaces $V$ and $W$ are isomorphic if there exists an isomorphism between them.

Exercise 2.26. Show the relation of being isomorphic is an equivalence relation.

Exercise 2.27. Show that an isomorphism maps bases to bases.

Theorem 2.28. If $\dim(V) = n$ then $V \cong \mathbb{R}^n$.

Proof: Choose a basis, $B$ of $V$, now map each vector to its coordinate vector, i.e. $v \mapsto [v]_B$.

Definition 2.29. We denote the image of $f$ as the set

$$\text{im}(f) = \{f(x) : x \in V\}$$

Definition 2.30. We denote the kernel of $f$ as the set

$$\ker(f) = \{x \in V : f(x) = 0\}$$

Exercise 2.31. For a linear map $f : V \to W$ show that $\text{im}(f) \leq W$ and $\ker(f) \leq V$.

Theorem 2.32. For a linear map $f : V \to W$ we have

$$\dim \ker(f) + \dim \text{im}(f) = \dim V.$$

Lemma 2.33. If $U \leq V$ and $A$ is a basis of $U$ then $A$ can be extended to a basis of $V$.

Exercise 2.34. Prove Theorem 2.32. HINT: apply Lemma 2.33 setting $U = \ker(f)$.
5 Vector spaces over number fields

**Definition 2.35.** A subset $F \subseteq \mathbb{C}$ is a number field if $F$ is closed under the four arithmetic operations, i.e. for $\alpha, \beta \in F$

(a) $\alpha + \beta \in F$
(b) $\alpha \beta \in F$
(c) $\frac{\alpha}{\beta} \in F$ (assuming $\beta \neq 0$).

**Exercise 2.36.** Show that if $F$ is a number field then $\mathbb{Q} \subseteq F$.

**Exercise 2.37.** Show that $\mathbb{Q}[\sqrt{2}]$ is a number field.

**Exercise 2.38.** Show that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} + c\sqrt{2^2} : a, b, c \in \mathbb{Q}\}$ is a number field.

**Exercise 2.39 (Vector Spaces over Number Fields).** Convince yourself that all of the things we have said about vector spaces remain valid if we replace $\mathbb{R}$ and $F$.

**Exercise 2.40.** Show that if $F, G$ are number fields and $F \subseteq G$ then $G$ is a vector space over $F$.

**Exercise 2.41.** Show that $\dim_{\mathbb{R}} \mathbb{C} = 2$.

**Exercise 2.42.** Show that $\dim_{\mathbb{Q}} \mathbb{R}$ has the cardinality of “continuum,” that is, it has the same cardinality as $\mathbb{R}$.

**Exercise 2.43 (Cauchy’s Equation).** We consider functions $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy’s Equation: $f(x + y) = f(x) + f(y)$ with $x, y \in \mathbb{R}$. For such a function prove that

(a) If $f$ is continuous then $f(x) = cx$.
(b) If $f$ is continuous at a point then $f(x) = cx$.
(c) If $f$ is bounded on some interval then $f(x) = cx$.
(d) If $f$ is measurable in some interval then $f(x) = cx$.
(e) There exists a $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) \neq cx$ but $g(x + y) = g(x) + g(y)$. (Hint: Use the fact that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. Use a basis of this vector space. Such a basis is called a Hamel basis.)

**Exercise 2.44.** Show that $1, \sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

**Exercise 2.45.** Show that $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}$ and $\sqrt{30}$ are linearly independent over $\mathbb{Q}$. 

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Exercise 2.46. * Show that the set of square roots of all of the square-free integers are linearly independent over $\mathbb{Q}$. (An integer is square free if it is not divisible by the square of any prime number. For instance, 30 is square free but 18 is not.)

Definition 2.47. A rational function $f$ over $\mathbb{R}$ is a fraction of the form

$$f(x) = \frac{g(x)}{h(x)}$$

where $g, h \in \mathbb{R}[x]$ (that is $g, h$ are real polynomials) and $h(x) \neq 0$ ($h$ is not the identically zero polynomial). More precisely, a rational function is an equivalence class of fractions of polynomials, where the fractions $\frac{g_1(x)}{h_1(x)}$ and $\frac{g_2(x)}{h_2(x)}$ are equivalent if and only if $g_1 \cdot h_2 = g_2 \cdot h_1$. (This is analogous to the way fractions of integers represent rational numbers; the fractions $3/2$ and $6/4$ represent the same rational number.) We denote the set of all rational functions as $\mathbb{R}(x)$.

Note that a rational function is not a function; it is an equivalence class of formal quotients.

Exercise 2.48. Prove that the rational functions $\{\frac{1}{x-\alpha} : \alpha \in \mathbb{R}\}$ are linearly independent set over $\mathbb{R}(x)$.

Corollary 2.49. $\dim_{\mathbb{R}[x]} \mathbb{R}(x)$ has the cardinality of “continuum” (the same cardinality as $\mathbb{R}$).

6 Elementary operations

Definition 2.50. The following actions on a set of vectors $\{v_1, \ldots, v_k\}$ are called elementary operations:

(a) Replace $v_i$ by $v_i - \alpha v_j$ where $i \neq j$.

(b) Replace $v_i$ by $\alpha v_i$ where $\alpha \neq 0$.

(c) Switch $v_i$ and $v_j$.

Exercise 2.51. Show that the rank of a list of vectors doesn’t change under elementary operations.

Exercise 2.52. Let $\{v_1, \ldots, v_k\}$ have rank $r$. Show that by a sequence of elementary operations we can get from $\{v_1, \ldots, v_k\}$ to a set $\{w_1, \ldots, w_k\}$ such that $w_1, \ldots, w_r$ are linearly independent and $w_{r+1} = \cdots = w_k = 0$.

Consider a matrix. An elementary row-operation is an elementary operation applied to the rows of the matrix. Elementary column operations are defined analogously. Exercise ?? shows that elementary row-operations do not change the row-rank of $A$.

Exercise 2.53. Show that elementary row-operations do not change the column-rank of a matrix.

Exercise 2.54. Use Exercises ?? and ?? prove the “amazing” Theorem ??.