1 Spectral Theorem

Definition 9.1. The Vandermode Matrix $V_n(x_1, \cdots, x_n)$ is defined to be the matrix

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^n & x_2^n & \cdots & x_n^n
\end{pmatrix}
$$

The terms $x_1, \cdots, x_n$ are called the generators of $V_n(x_1, \cdots, x_n)$.

Exercise 9.2. If $x_1, \ldots, x_n$ are distinct, then $V_n$ has full rank.

Exercise 9.3. $\det(V_n(x_1, \ldots, x_n)) = \prod_{1 \leq j < i \leq n}(x_i - x_j)$

Theorem 9.4. The identity permutation cannot be represented as the product of an odd number of transpositions.

Definition 9.5. A matrix $A$ (with coefficients in $\mathbb{R}$) is a real symmetric matrix if $A = A^T$, i.e. if $A = (\alpha_{ij})$ then $\alpha_{ij} = \alpha_{ji}$.

Remark 9.6. Over $\mathbb{C}$, every matrix has an eigenvalue (by the Fundamental Theorem of Algebra.)

Theorem 9.7. All eigenvalues of a real symmetric matrix are real.

Proof: Suppose $Ax = \lambda x$, $x \neq 0$. We want to show that $\lambda = \overline{\lambda}$. If we let $x^*$ denote $x^T$, then

$$
x^*Ax = x^*(\lambda x) = \lambda x^*x = \lambda(x_1^*x_1 + \ldots + x_n^*x_n) = \lambda \left( \sum_{i=1}^{n} |x_i|^2 \right)
$$
and so
\[ \lambda \left( \sum_{i=1}^{n} |x_i|^2 \right) = x^*Ax = x^*Ax = (x^*Ax)^* = \left( \lambda \sum_{i=1}^{n} |x_i|^2 \right)^* = \bar{\lambda} \left( \sum_{i=1}^{n} |x_i|^2 \right) \]
which implies \( \lambda = \bar{\lambda} \).

**Example 9.8.** Let
\[ A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \]

Then \( A \) is a symmetric matrix with characteristic polynomial \( f_A(x) = x^2 - (\text{tr}(A))x + \det(A) = x^2 -(a+d)x + (ad - b^2) \), which has real roots if and only if the discriminant is nonnegative. This is true because the discriminant is:
\[ (a + d)^2 - 4(ad - b^2) = a^2 - 2ad + d^2 + 4b^2 = (a - d)^2 + 4b^2 \geq 0 \]

Recall the following fact:

**Theorem 9.9.** A matrix \( A \) with coefficients in a field \( \mathbb{F} \) is diagonalizable if and only if \( A \) has an eigenbasis, i.e. \( \mathbb{F}^n \) has a basis consisting of eigenvectors of \( A \).

**Definition 9.10.** The inner product (on \( \mathbb{R}^n \)) is a function \( \cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) defined as
\[ \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i \beta_i = \mathbf{a}^T \mathbf{b} \]
where
\[ a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \]

**Definition 9.11.** The (Euclidean) norm of a vector \( \mathbf{a} \), written \( \| \mathbf{a} \| \), is defined to be \( \sqrt{\mathbf{a} \cdot \mathbf{a}} \).

**Definition 9.12.** The set of vectors \( \{ \mathbf{a}_1, \mathbf{a}_2, \cdots , \mathbf{a}_k \} \) is orthonormal (ON) iff
1. \( \mathbf{a}_i \perp \mathbf{a}_j \), i.e. \( \mathbf{a}_i^T \mathbf{a}_j = 0 \), for all \( i \neq j \)
2. \( \| \mathbf{a}_i \| = 1 \), i.e. \( \mathbf{a}_i^T \mathbf{a}_i = 1 \), for all \( i \).

**Exercise 9.13.** If \( \mathbf{a}_1, \cdots , \mathbf{a}_k \) are orthonormal, then they are linearly independent.

**Definition 9.14.** An orthonormal basis (ONB) is an orthonormal set of \( n \) vectors \( \mathbf{a}_1, \mathbf{a}_2, \cdots , \mathbf{a}_n \in \mathbb{R}^n \) (which thus forms a basis).
Theorem 9.15 (Spectral Theorem). If $A$ is a symmetric real matrix, then $A$ has an orthonormal eigenbasis.

Observation 9.16. The standard basis on $\mathbb{R}^n$ is orthonormal. If we want to change to another ONB $\{e_1, \ldots, e_n\}$, then our base-change matrix would be:

$$S = \begin{pmatrix} e_1 & \ldots & e_n \end{pmatrix}$$

so that

$$ST = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

and so $STS = I$ because $e_i^T e_j = 0$ for $i \neq j$, and $e_i^T e_i = 1$ for all $i$ (by orthonormality.)

Observation 9.17. The following are equivalent for an $n \times n$ matrix $S$.

1. The columns of $S$ are orthonormal.
2. $STS = I$

Recall the following:

Theorem 9.18 (Amazing Fact 1). $\text{rk}(S) = \text{rk}(ST)$.

Our last observation leads us to the following:

Theorem 9.19 (Amazing Fact 2). If the columns of a matrix $S$ are orthonormal, then so are the rows.

Proof: The columns of $S$ are ON $\Rightarrow S^T S = I \Rightarrow ST = S^{-1} \Rightarrow S S^T = I \Rightarrow (ST)^T ST = I \Rightarrow$ the columns of $ST$ are ON $\Rightarrow$ rows of $S$ are ON.

Definition 9.20. A real $n \times n$ matrix is an orthogonal matrix iff its transpose is its inverse.

Theorem 9.21 (Spectral Theorem Restated). If $A$ is a real symmetric matrix, then there is an orthogonal matrix $S$ such that $S^{-1}AS = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$ are real numbers (and thus the eigenvalues of $A$.)

Definition 9.22. Two matrices $A, B$ are similar (written $A \sim B$) iff there is a matrix $S$ such that $B = S^{-1}AS$.

Observation 9.23. If $A$ is a symmetric real matrix, then $A$ has a real eigenvector $\frac{f_1}{||f_1||}$, where w.l.o.g. we can take $||f_1|| = 1$.

Definition 9.24. Suppose $U \subseteq V$, i.e. $U$ is a subspace of $V$, and $\varphi : V \to V$ is a linear map. We say that $U$ is an invariant subspace for $\varphi$ iff $\varphi(U) \subseteq U$, i.e. for all $u \in U$, $\varphi(u) \in U$. 

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Theorem 9.25. Let $A$ be a nonsingular matrix, and suppose $U$ is a right-invariant subspace of $A$, i.e. for all $u \in U$, $Au \in U$. Then $U^\perp$ is a left-invariant subspace of $A$ (i.e. $U^\perp$ is a right-invariant subspace of $A^T$).

Corollary 9.26. Suppose $A$ is a symmetric real matrix. If $U$ is an invariant subspace for the linear map determined by $A$ (i.e. $U$ is a left-invariant subspace of the matrix $A$), then so is $U^\perp$. 