Discrete Math, 1st day, Monday 6/21/04
REU 2004. Info: http://people.cs.uchicago.edu/∼laci/reu04.

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1 Points in the plane

Exercise 1.1 (Sylvester). Given n points in the plane, not all on the same line, show that there is a line in the plane that passes through exactly two of them. What if we switch the role of points and lines, i.e. given n lines in the plane such that not all of them pass through the same point, is it true that there is a point that belongs to exactly two of them? What if in addition, no two of the lines are parallel?

Exercise 1.2 (Erdős-E. Klein). Show that there is a constant c, independent of n, such that given n points in the plane, the number of pairs of points with distance 1 is less than $cn^{3/2}$.

Note: It is known that this number is in fact less that $cn^{5/4}$ (for possibly a different constant c).

OPEN QUESTION: Is it less than $cn^{1.1}$?

2 Games

Exercise 1.3. Let n be even, and consider an $n \times n$ chessboard, with two opposite corners removed. Show that it cannot be covered with dominoes. (A domino covers two neighboring squares, and dominoes don't overlap.) (An "Aha" problem.)

Exercise 1.4 (John Conway). Consider an infinite chessboard: that means, a chessboard with infinitely many squares in all directions. We will draw one line which cuts the chessboard into to infinite halves, the North and the South half. This could be viewed as the coordinate plane where the integer lattice (i.e. points (a, b) for a and b integers) are the corners of squares, and the x-axis is the **equator** separating the board into the North and South parts. The North side has some army of pieces, all identical, with at most one on each square, and the South side is empty. North's goal is to take its pieces and invade as deep as possible into South's territory. To do this, the only move allowed is to take two of North's pieces which are adjacent (possibly diagonally adjacent) and have one jump (or step) over the other piece, thereby killing the second piece, ending up across on the other side. In other words, two adjacent pieces can be replaced with a third piece which makes a line of three adjacent pieces with the first two. Prove that North cannot invade too far into South's territory: say, prove that North cannot advance beyond 100 steps south of the equator. (Note that North may have begun with an infinite number of pieces.) (Note: 8 is the maximum depth.)

Exercise 1.5 (Paul Erdős). a) Take $n+1$ numbers between 1 and $2n$, inclusive. (assigned to Lajós Pósa at Pósa's age of 12). Prove that two of them are relatively prime. (an "aha" problem)
b) Again take $n+1$ numbers between 1 and $2n$, inclusive. Prove that one is a factor of another. (an "ahaaa" problem)

Exercise 1.6. Take a rectangular pool table, and a large collection of pennies, all round of the same radius (assume that the pool table fits at least one penny). Suppose two people take turns, each turn consisting of placing another penny on the pool table so that it does not overlap with any other penny and does not dangle off the side. The first player who cannot fit a penny on the board loses. Prove that the first player has a winning strategy.

Divisor Game. Fix an integer $n > 1$. Player 1 starts the game by picking a divisor of n. Player 2 picks another divisor of n that is not a divisor of the divisor picked by Player 1. Play alternates between the two players with each player choosing a divisor of n that is not a divisor of any divisor already picked. Eventually, one player is forced to choose n, and that player loses. A winning strategy for Player 1 was found in class for $n = 30, 12$. For primes p, q, and r, we generalized winning strategies for $n = pqr, p^k, p^kq$.

Exercise 1.7. Find a winning strategy for Player 1 when $n = p^2q^2$.

Exercise 1.8. Prove that Player 1 has a winning strategy for all $n \geq 2$. (*Hint:* Prove the *existence* of a winning strategy. No explicit strategy is known that works for all n.)

On an $n \times n$ chessboard, we can "infect" any initial configuration of cells. Then the infection spreads: a cell becomes infected if at least two adjacent cells are infected (diagonal neighbors do not count). How few cells can we initially infect so that the whole board becomes infected? Note that if we infect the n cells on a diagonal, the whole board becomes infected.

Exercise 1.9. Show that in order to get the entire board infected, we need at least n initially infected cells.

3 Ramsey Numbers

Ramsey Game. Player 1 has a blue pencil, and Player 2 has a red pencil. Given n points in the plane, no three on a line, Player 1 starts by connecting a pair of points with a blue line. Player 2 connects a different pair with a red line. The players continue taking turns by assigning their color to an uncolored pair of points. Player 1 loses if a blue triangle is formed, that is three of the n points are connected to each other with blue lines. Similarly, Player 2 loses if a red triangle is formed.

For the game $n = 5$, it is possible that no player loses: color the five sides of a regular pentagon blue, and color the five diagonals red. In this way, no triangles are made. We will see that this is not possible in the $n = 6$ game, but first we introduce some notation.

Definition 1.10 (Erdős-Rado Arrow Notation). The notation $n \to (k, \ell)$ means that for any red and blue coloring of the $\binom{n}{2}$ pairs of n points, there is either a subset of k points all of whose pairs are colored red or a subset of ℓ points all of whose pairs are colored blue.

We saw above that $5 \not\to (3, 3)$ and we prove

Theorem 1.11.

$$6 \to (3, 3)$$

Proof. Select one point, call it 1. Since 1 is paired with five other points, at least three of these pairings are the same color, say blue. Denote these three other numbers by $2, 3$, and 4. If a pair among $\{2, 3, 4\}$ is also blue, say $\{2, 4\}$, then $\{1, 2, 4\}$ is a blue triangle. If no pair among $\{2, 3, 4\}$ is blue, then all three pairs are red, so $\{2, 3, 4\}$ is a red triangle.■

Notice that the arrow notation is more general than the Ramsey game because the arrow notation does not put any restrictions on the proportion of blue or red pairs. In particular, we found a coloring in the case $n = 6$ so that there are 9 red pairs but no red triangles (the three houses, three utilities example).

Exercise 1.12. Show $10 \to (3, 4)$.