Discrete Math, 2nd day, Tuesday 6/22/04
REU 2004. Info:
http://people.cs.uchicago.edu/∼laci/reu04.

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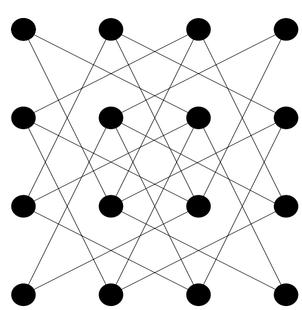
1 Games continued

Exercise 2.1 (Dominoes). Prove: if we remove two opposite corners from the chessboard, the board cannot be covered by dominoes. (Each domino covers two neighboring cells of the chessboard.) Look for an "Ah-ha" proof: clear, convincing, no cases to distinguish.

Exercise 2.2 (Triominoes). Remove a corner from a 101×101 chessboard. Prove that the rest cannot be covered by **triominoes**. A triomino is like a domino except it consists of three squares in a row; each cell can cover one cell on a chessboard. Each triomino can either "stand" or "lie." Look for an "Ah-ha" proof.

Exercise 2.3 (Knight's trail). Consider a knight moving around on a 4×4 chessboard. We let the knight start at any cell of our choosing, and we wish to guide it through 15 moves so it never steps on a previously visited cell. So, after the 15 moves, the knight will have visited each cell. Prove that this is impossible. Find an "Ah-ha" proof.



Graph of knight moves on a 4×4 chessboard.

Hint: Show that from any trail of the knight's moves, if you delete 4 cells, the trail splits into no more than 5 connected parts. (A set S of cells is **connected** from the knight's point of view if the knight can move from any cell of S to any other cell of S without leaving S.)

Exercise 2.4. A mouse finds a $3 \times 3 \times 3$ chunk of cheese, cut into 27 blocks (cubes), and wishes to eat one block per day, always moving from a block to an adjacent block (a block that touches the previous block along a face). Moreover, the mouse wants to leave the center cube last. Prove that this is impossible. Find two "Ah-ha!" proofs; one along the lines of the solution of knight's trail problem, the other inspired by the solution of the dominoes problem.

Exercise 2.5. 96 kids wait for us to split an 8×12 chocolate bar along the grooves into 96 small rectangles. It is up to us in what order we do the splitting; we can start, for instance, by breaking the 7 long grooves and then split each of the 8 long (1×12) pieces; or we can start with the short grooves, or halve the bar each time, or any other way. The one thing we are not permitted to do is stack the pieces. At any one time, we have to pick up one piece and break it into two.

Each break takes us 1 second. Find the fastest method. (This is another "Ah-ha" problem.)

Exercise 2.6. n teams play a straight-elimination tournament; there are no ties. How many matches do they need to play before the winner is declared? (An "Ah-ha" problem again.)

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Study the handout for the basic definitions involving graphs, including complete graphs, (complete) bi(multi-)partite graphs, (induced) subgraphs, the degree of a vertex, complement of a graph and Hamilton cycles. In particular, recall that an isomorphism from the graph $\mathcal{G} = (V, E)$ to the graph $\mathcal{H} = (W, F)$ is a bijection $f : V \to W$ between the vertices that preserves the adjacency relation, i.e. for any two vertices $x, y \in V$, x, y are adjacent in $\mathcal{G}$ iff (if and only if) $f(x), f(y)$ are adjacent in $\mathcal{H}$. $\mathcal{G}$ is said to be isomorphic to $\mathcal{H}$ if there exists an isomorphism from $\mathcal{G}$ to $\mathcal{H}$.

Exercise 2.7. Show that isomorphism of graphs is a transitive relation.

Exercise 2.8. Show that the number of functions $f : A \to B$ is $|B|^{|A|}$.

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We defined **decision tree** and the related concepts of **root, node, leaves, parent,** and **child.** For example, flipping n coins can be made into a decision tree with 2^n leaves. Similarly, we can make a decision tree from the moves in a chess game, but the tree is more complicated. For example, the leaves will not all be at the same level since different games end after different numbers of moves. Furthermore, the possible moves at each node are dependent on the preceding moves or the **history.** Although the decision tree for a chess game is far too large to store in a computer (it has more configurations than the number of atoms in the earth), such trees can on principle be analyzed in a finite time for a winning strategy.

First, assign a value of B, W, or D to the leaves of the tree if the outcome is a black win, a white win, or a draw, respectively. The nodes above the leaves can be assigned a value of B, W, or D if there is an optimal strategy for black to win, white to win, or a draw. Suppose the node x is a white move. Then the value of x is W if $\exists$ W child of x, B if all children are B, and D if there is no W child and $\exists$ D child.

Definition 2.9. A **strategy function** for white is a function from the set of histories preceding a white move to the set of possible next moves. A **winning** strategy function for white makes moves to nodes with value W.

If a winning strategy exists for white, then the root must have value W, and there is a W child all of whose children are W. Each of these children must have a W child all of whose children are W, etc.

Theorem 2.10. *If a finite game only has win or lose outcomes, then one of the players has a winning strategy.*

This theorem can be used to solve the Divisor Game problem. (*Hint:* Proof by contradiction.)

Exercise 2.11. Exercise 6.1.5 p. 44 of the notes.

2 Hamilton cycles

Definition 2.12. A **Hamilton cycle** is a subgraph that is an n-cycle in a graph on n vertices. A graph is **Hamiltonian** if it has a Hamilton cycle.

Exercise 2.13. The $k \times \ell$ grid is Hamiltonian if and only if $k\ell$ is even and $k,\ \ell \geq 2$.

Method 1. Make the vertices of the grid into the cells of a chess board. *Hint:* How many steps do you move to get back to your starting color? How many steps do you move to get all the cells?

Method 2. *Sam's hint:* Use the lemma:

Lemma 2.14. *If G is Hamiltonian then removing k vertices splits G into at most k connected components.*

Definition 2.15. A graph G is bipartite if its vertices can be colored red and blue such that adjacent vertices have different colors.

Observe: if G is bipartite and Hamiltonian, then the two parts are equal. Exercise **??** is a consequence of this.

Method 3. *Alex's hint:* What goes up, must come down. What moves left, must return to the right.

Exercise 2.16 (Dirac). Prove: if every vertex in a graph $\mathcal{G}$ of n vertices has degree $\geq n/2$, then $\mathcal{G}$ is Hamiltonian (i.e. it contains a Hamiltonian cycle).

Exercise 2.17. Let $\mathcal{G}$ be a graph with n vertices and m edges. If $m \geq \binom{n-1}{2} + 2$ then $\mathcal{G}$ is Hamiltonian.

Definition 2.18. A graph $\mathcal{G}$ is bipartite if the set of vertices V partitions into two disjoint subsets, V_1 and V_2 (i.e. so that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) such that no vertex of V_1 is adjacent to any other vertex of V_1, and no vertex of V_2 is adjacent to any other vertex of V_2. In other words, all edges of $\mathcal{G}$ connect one vertex from V_1 and one vertex from V_2.

In other words, a graph is bipartite iff it is a subgraph of a complete bipartite graph (cf. p. 44 of the notes).

Exercise 2.19. Prove that a graph $\mathcal{G}$ is bipartite iff (if and only if) $\mathcal{G}$ has no odd cycle.

Exercise 2.20. Exercise 6.1.10, p. 45 of the notes.

Exercise 2.21. Exercise 6.1.11, p. 45 of the notes.