1 Graph theory

**Exercise 3.1.** Let $R$ be an equivalence relation on a set $\Omega$. Prove that we can obtain from $R$ a (unique) partition of $\Omega$ (such that the partition induces the original equivalence relation in the natural way).

**Exercise 3.2.** Recall: Every connected graph contains a spanning tree. Prove this by induction on the number of edges (i.e., complete the proof given in class). Recall that if we remove an edge from a cycle in a connected graph, the graph remains connected, and that this decreases the (finite) number of cycles (cycles have no repeated vertices).

**Exercise 3.3.** (A)

(a) In every tree, there exists a vertex that is common to all the longest paths (i.e., paths of greatest length).

(b) In every tree, if the length of a longest path is odd then there is an edge common to all the longest paths.

(c)* Disprove (a) for connected graphs.

**Exercise 3.4.** (B)

(Helly-type Theorem) Let $R_1, \ldots, R_k$ be subtrees of a tree, and suppose that for all $i, j$ we have $V(R_i) \cap V(R_j) \neq \emptyset$. Then $V(R_1) \cap \ldots \cap V(R_k) \neq \emptyset$.

**Exercise 3.5.** (C)

Let $G$ be a connected graph. Let $P_1, P_2$ be longest paths in $G$. Prove that $V(P_1) \cap V(P_2) \neq \emptyset$.

**Note:** (C) and (B) imply (A).
Definition 3.6. A legal coloring of a graph \( G \) is an assignment of a “color” to each vertex such that adjacent vertices have different “colors”.

The chromatic number \( \chi(G) \) of \( G \) is defined to be the minimum number of colors necessary to color \( G \) legally.

Exercise 3.7. The graph \( G \) is bipartite if and only if \( \chi(G) \leq 2 \).

Exercise 3.8. Construct a graph \( G \) such that \( K_3 \not\subseteq G \) and \( \chi(G) = 4 \).

Hint: 11 vertices, 5-fold symmetry.

2 Inequalities

For a set of \( n \) positive real numbers, \( x_1, \ldots, x_n \) we can define several means:

1. the arithmetic mean \( A(x_1, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \);

2. the quadratic mean \( Q(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} = A(x_1^2, \ldots, x_n^2) \);

3. the geometric mean \( G(x_1, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \);

4. the harmonic mean \( H(x_1, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} = A\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \).

Exercise 3.9. Prove, for any \( x_1, \ldots, x_n \), the inequalities

\[
Q(x_1, \ldots, x_n) \geq A(x_1, \ldots, x_n) \geq G(x_1, \ldots, x_n) \geq H(x_1, \ldots, x_n).
\]

Hint: \( Q \geq A \) is easy; \( A \geq G \) is difficult; \( G \geq H \) follows immediately from \( A \geq G \).

Exercise 3.10 (JENSEN’S INEQUALITY). There is a generalization of all of the above inequalities: the discrete version of Jensen’s inequality. Let \( I \subseteq \mathbb{R} \) be a finite or infinite interval and let \( f : I \to \mathbb{R} \) be a function. If \( f \) satisfies the inequality

\[
(\forall x, y \in I) f\left(\frac{x + y}{2}\right) \geq \frac{f(x) + f(y)}{2},
\]

then

\[
f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{f(x_1) + \cdots + f(x_n)}{n},
\]

i.e.

\[
f(A(x_1, \ldots, x_n)) \geq A(f(x_1), \ldots, f(x_n)).
\]

Similarly, if:

\[
(\forall x, y \in I) f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2},
\]

then

\[
f(A(x_1, \ldots, x_n)) \leq A(f(x_1), \ldots, f(x_n)).
\]
Note: A continuous function satisfying inequality (??) is concave and a continuous function satisfying inequality (??) is convex. However, not every function satisfying either (or even both) inequalities is continuous. (Prove!)

**Exercise 3.11.** Prove Jensen’s inequality. Hint: First show inductively that the inequality is true for $n = 2^k$ (that is, by induction on $k$). Then, show that the inequality for $2^k$ implies the inequality for any $n < 2^k$ as well, thus proving the inequality for all $n$. 

**Exercise 3.12.** Deduce Exercise ?? from Exercise ??.

### 3 Graphs without short cycles

We proved in class the

**Theorem 3.13 (Kővári-Sós-Turán).** (cf. Exercise 6.1.22 in the text) There exists $c > 0$ such that, if $G$ is a graph with $m$ edges and $n$ vertices, and $G$ has no 4-cycles, i.e. $G \not\cong C_4$, then $m \leq cn^{3/2}$. In other words, $m = O(n^{3/2})$.

In fact we showed that, as $m$ tends to infinity, $m \leq \frac{1}{2}n^{3/2}$.

The following exercise strengthens this result:

**Exercise 3.14.** Show that, in the above situation of $G$ with $m$ edges, $n$ vertices, s.t. $G \not\cong C_4$, that 

$$m \leq \frac{n^{3/2} + n}{2}$$  \hspace{1cm} (7)

Hint: use the following inequality from the proof of the theorem:

$$\frac{n - 1}{2} \geq \left(\frac{2m}{n}\right)$$  \hspace{1cm} (8)

The following exercise shows the tightness of the bound:

**Exercise 3.15.** Show that there exists $c' > 0$ such that for infinitely many choices of $n$, there exists a graph $G \not\cong C_4$ with $m$ edges and $n$ vertices, such that $m > c'n^{3/2}$.

**Exercise 3.16.** Show that $(\forall k)(\exists G \not\cong K_3), (\chi(G) = k)$. In words, there exist graphs which do not contain triangles which have arbitrarily high chromatic number.

**Exercise 3.17.** Show that $(\forall k, \ell)(\exists G$ such that $G$ has no odd cycles of length $\leq \ell$, and $\chi(G) = k)$. In words, show that for any odd $\ell \geq 1$, there exist graphs with arbitrarily high chromatic number which do not contain any odd cycles of length $3, 5, \ldots, \ell$.

**Remark 3.18.** Exercise ?? remains valid if we drop the word ”odd”:
Theorem 3.19 (Erdős 1960). \((\forall k, \ell)(\exists G \text{ such that } G \text{ has no cycles of length } \leq \ell, \text{ and } \chi(G) = k)\). In words, for any \(\ell \geq 1\), there exist graphs with arbitrarily high chromatic number which do not contain any cycles of length 3, 4, 5, \ldots, \ell.

Erdős proved this in 1960 using the probabilistic method but did not exhibit any specific examples of such graphs. No explicit construction was known of such graphs until 30 years later, around 1990.