1 Graph theory continued

**Exercise 4.1.** If $G$ is connected with $n \geq 2$ vertices, then there exists a vertex $v$ such that $G \setminus v$ is connected (the latter means the subgraph we get by throwing out the vertex $v$ and all edges incident with $v$). Note that we cannot pick just any vertex, as evidenced by the case of throwing out the center of a star (which leaves a highly disconnected graph).

**Definition 4.2.** A graph $G$ is $k$-edge-connected between vertices $x$ and $y$ if there are $k$ edge-disjoint paths between $x$ and $y$.

**Definition 4.3.** A graph $G$ is $k$-vertex-connected between vertices $x$ and $y$ if there are $k$ vertex-disjoint paths between $x$ and $y$.

Suppose we found 57-edge-disjoint paths between $x$ and $y$, and we think 57 is optimal. How can we demonstrate that 58 such paths cannot be found?

**Ben’s Conjecture:** If there are no more than 57 edge-disjoint paths from $x$ to $y$, then there is a partition $V = A \cup B$ such that $x \in A$, $y \in B$, and the maximum number of edges between $A$ and $B$ is less than or equal to 57.

**Definition 4.4.** An $(x, y)$-cut of a graph $G = (V, E)$ is a partition $V = A \cup B$ such that $x \in A$ and $y \in B$.

**Theorem 4.5 (Menger’s Theorem).** The maximum number of edge-disjoint paths between $x$ and $y$ equals the minimum number of edges between the two parts of an $(x, y)$-cut.

2 Puzzles

**Exercise 4.6.** If we have a $10^{10} \times 10^{10}$-size square that we want to cover with $1 \times 1$-squares, evidently we can fit $10^{20}$ squares. Show that if we have a $(10^{10} + 0.1) \times (10^{10} + 0.1)$-square then we can fit $10^{20} + 1$ squares.
Exercise 4.7. Suppose we have a big rectangle which is $a \times b$ in size, and we tile it with various size rectangles $a_n \times b_n$ (situated parallel to the sides of the original rectangle), where for each $n$, at least one of $a_n$ or $b_n$ is an integer. Show that either $a$ and $b$ is an integer.

3 Set game and Szemerédi’s theorem

Set Game. The Set game is played with a deck of cards. Each card has four properties with three possible values: a shape (oval, diamond, squiggle), a color (red, green, purple), a shading (blank, solid, shaded), and a number (1, 2, 3). Each card is distinct, so there are $3^4 = 81$ cards.

Twelve cards are placed face up on a table. The objective is to find a “set” among the twelve cards. A “set” is formed by three cards so that each property is either the same on all cards or different on all cards. The first player to find a “set” gets that “set,” after which three new cards from the deck are placed face up to replace them. The player who finds the most “sets” wins.

We can generalize the Set game so that there are $n$ properties on each card, and each property has three possible values. In this $n$-dimensional Set game we have $3^n$ cards. We are interested in how many cards we can draw from the deck without finding a “set.” Let $\alpha(n)$ denote the maximum number.

Let $\mathbb{Z}_3$ be the set $\{0, 1, 2\}$ with addition mod 3, so $2 + 2 = 1$ for example. Let $\mathbb{Z}_3^n$ be the set $\{(x_1, \ldots, x_n) : x_i \in \mathbb{Z}_3\}$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{Z}_3^n$, we define addition by $x + y = (x_1 + y_1, \ldots, x_n + y_n)$. In the Set game ($n = 4$), the cards correspond to the quadruples in $\mathbb{Z}_3^4$; for instance [oval, green, blank, 3] $\leftrightarrow [1, 1, 0, 2]$.

Observation 4.8. Three distinct elements $x, y, z \in \mathbb{Z}_3^4$ form a set if and only if $x + y + z = 0$.

Similarly, we can work in $\mathbb{Z}_3^n$ for the $n$-dimensional Set game. So an alternative definition for $\alpha(n)$ is the maximum number of elements of $\mathbb{Z}_3^n$ without a solution to $x + y + z = 0$ where $x, y, z$ are distinct.

Definition 4.9. A function $f : \mathbb{N} \to \mathbb{R}$ is supermultiplicative if $f(n + m) \geq f(n) \cdot f(m)$.

Exercise 4.10. Prove: if $f$ is supermultiplicative, then $L = \lim_{n \to \infty} \sqrt[f(n)]{n}$ exists, and $L \geq \sqrt[f(n)]{n}$ for all $n$.

Exercise 4.11. Prove $\alpha(n)$ is supermultiplicative.

Observe that $2^n \leq \alpha(n) \leq 3^n$, so $2 \leq L \leq 3$ by the previous two exercises, where $L = \lim_{n \to \infty} \sqrt[n]{\alpha(n)}$.

Exercise 4.12. Prove: $\alpha(4) \geq 20$. Infer: $L \geq \sqrt[20]{20} \approx 2.115$.

Exercise 4.13. Prove: $\alpha(2) = 4$. Infer $\alpha(4) \leq 36$. 
Exercise 4.14. Prove: $\alpha(3) = 9$. Infer $\alpha(4) \leq 27$.

Exercise 4.15. Prove: $\alpha(4) \leq 24$.


Conjecture 4.17 (Open Problem). $\lim_{n \to \infty} \sqrt[\alpha(n)]{n} = 3$.

Exercise 4.18. Color the integers red and blue. Is there a set of 3 red or 3 blue integers that form an arithmetic progression? Find a coloring of the positive integers such that there is no arithmetic sequence of 3 red or 3 blue integers in the first $n$ numbers. What is the largest $n$ for which you can find an example?

Theorem 4.19 (Van der Waerden, 1928). If we color the positive integers with $k$ colors, then for all $\ell$ there is an $\ell$-term arithmetic progression in one color.

Theorem 4.20 (Szemerédi, 1975). For all $\epsilon > 0$ and $\ell$, there is an $N$ such that if $A \subseteq \{1, \ldots, N\}$ and $|A| \geq \epsilon N$, then $A$ contains an $\ell$-term arithmetic progression.

Exercise 4.21. Prove that Szemerédi’s Theorem implies van der Waerden’s Theorem.

History:

- Erdős and Turán conjectured this result in 1936.
- In 1952, K. F. Roth found an analytic proof of the result for $\ell = 3$. This result, in addition to another major result, were part of his Fields Medal citation in 1958.
- In the late 1960s, E. Szemerédi found a combinatorial proof for $\ell = 3$.
- Later, E. Szemerédi found a combinatorial proof for $\ell = 4$.
- In 1975, E. Szemerédi found a combinatorial proof of the whole conjecture.
- Subsequently, Furstenberg found an analytic proof of Szemerédi’s Theorem. His methods led to further generalizations. For example:

Theorem 4.22. $\alpha(n) = o(3^n)$.

On April 8, 2004, a major breakthrough was announced in a very old problem that has fascinated mathematicians for ages:

Theorem 4.23 (Green-Tao, 2004). For all $\ell$, there is an $\ell$-term arithmetic progression among prime numbers.

Ben Green and Terence Tao announced this result in their paper “The Primes Contain Arbitrarily Long Arithmetic Progressions.” The proof uses Szemerédi’s Theorem and the ideas of Furstenberg’s proof. It is posted on arXiv.