Discrete Math, 7th day, Wednesday 6/30/04
REU 2004. Info:
http://people.cs.uchicago.edu/∼laci/reu04.

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Puzzle 7.1. (a) A 3-way lamp (red,yellow,green) is connected to n 3-way switches. Each configuration of the switches corresponds to a state of the lamp. In other words we have a function $f : [3]^n \to [3]$. It is known that if we change all the switches, then the lamp will switch also. Show that all but one of the switches are dummies. In other words, the function f depends on only one of the variables: one switch determines the state of the lamp.

(b) Show that this is not necessarily true if we have infinitely many switches. In fact, we can arrange it so that the lamp does not change whenever we change finitely many switches. (But it does change if we change all of them.)

Puzzle 7.2. A hundred prisoners are on their way to a prison island. They know a lot about the workings of the prison. First of all, the moment they arrive they will be taken to individual cells, never to see or talk to each other again. Also, there is a special room that they are taken to, where they are being interrogated by the warden. They are taken there one at a time, but in unknown order. They have no way of knowing who else has entered the room before them. They know that each one of them will be called to the room infinitely many times, but they don't know in what order. In the room there is a two-way switch (up,down), which they get to see and are allowed to change if they want to. Nobody other than the prisoners will ever touch the switch. There is also a red button, which has an amazing property: If one of the prisoners presses it and by that time all the prisoners have been in this room at least once, then the game is over and they all go free. If one of them hasn't, then they all spend the rest of their lives in prison. (By the way, the prisoners are immortal, so this will be a really long time.) The prisoners have a lot of time before they get to the prison to make their plans. Find a way for them to get out.

There are two versions to this problem. In the easier version, the prisoners know the initial position of the switch. In the other they don't. Figure out a strategy for each problem.

1 Fishermen's clubs

In a small fishing town by a lake, the n fishermen of the town are in the habit of forming as many clubs as they can. To control the situation a bit the state has formed a rule that any two clubs must share *exactly* one member. The question is raised, how many clubs can there be. It is easy to come up with n clubs. For instance we could have one really passionate person, who has formed $n-1$ clubs, one with each other fisherman. (So all these clubs so far have exactly 2 members each.) Then all the other fishermen, having gotten tired of him, decide to form a club, for a total of n clubs. Or maybe he was faster than them and formed a club by himself. Notice though that not both of those last clubs can be formed, because they don't share a member. Another way to form n clubs is if $n = k^2 + k + 1$ and there is a projective plane of order k; the points correspond to the fishermen and the lines to the clubs. The natural question then is whether we could have more than n clubs. The surprising answer is given by the following theorem:

Theorem 7.3 (Erdős-deBruijn, 1949). *Given a set with n elements, there cannot be more than n subsets with the property that any two of them share exactly one member.*

A reasonable next step would be to ask: What if we allow them to have exactly k members in common? The answer remains the same:

Theorem 7.4 (Fisher-R.C.Bose-Majumdar). *Given $k \geq 1$ and given a set with n elements, there cannot be more than n subsets with the property that any two of them share exactly k members.*

The proof is not hard but is quite ingenious: it invokes an unexpected tool, linear algebra. The method, introduced by R.C. Bose, has since produced volumes of startling results.

Definition 7.5. (a) The **characteristic vector** (**incidence vector**) of a subset $A \subseteq [n]$ is the vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$, where $x_i = 1$ if $i \in A$ and $x_i = 0$ if $i \notin A$.

(b) If $A_1, \ldots, A_k$ are subsets of $[n]$, the **incidence matrix** of this set-system (hypergraph) is the $k \times n$ matrix whose i-th row is the characteristic vector of A_i.

Exercise 7.6 (R.C. Bose, 1949). Let $k \geq 1$ and let $A_1, \ldots, A_k \subseteq [n]$. Show that if $(\forall i \neq j)(|A_i \cap A_j| = k)$, the characteristic vectors of the A_i are linearly independent.

Since there cannot be more than n linearly independent vectors in $\mathbb{R}^n$ ("Fundamental Fact of Linear Algebra"), this proves Theorem **??**.

(Note: Bose proved this for uniform hypergraphs; Majumdar extended the result to the nonuniform case.)

2 Ramsey Theory

Notation: If A is a set then $\binom{A}{k}$ denotes the set of all k-subsets (subsets of size k) of A. So if $|A| = n$ then $\left|\binom{A}{k}\right| = \binom{n}{k}$. Recall that $[n] = 1, \ldots, n$.

Definition 7.7 (Erdős-Rado arrow notation). $(n)^r_k \to (s_1, \ldots, s_k)$ if for any partition of $\binom{[n]}{r} = A_1 \dot{\cup} \ldots \dot{\cup} A_k$ there exists i, $1 \le i \le k$ and $H \subseteq [n]$ such that $|H| \ge s_i$ and $\binom{H}{r} \subseteq A_i$. Absence of a superscript on (n) indicates that $r = 2$. The subscript is redundant and often omitted.

Theorem 7.8 (Ramsey). *For all $r, k, s_1, \ldots, s_k$ there exists n such that $(n)^r_k \to (s_1, \ldots, s_k)$.*

Notation: $R_r(s_1, \ldots, s_k)$ denotes the smallest such n and is called a **Ramsey number**. If $r = 2$ then r is omitted.

Exercise 7.9. Prove that $17 \to (3, 3, 3)$.

We have $6 \to (3, 3)$ and $17 \to (3, 3, 3)$. In general, $\lceil n!e \rceil \to (3, \ldots, 3)$ (n parts, i. e., $(\lceil n!e \rceil)^2_n \to (3, \ldots, 3)$.

Here are some known Ramsey numbers.

$6 \to (3, 3), 5 \not\to (3, 3)$, so $R(3, 3) = 6$.

$10 \to (3, 4), 9 \not\to (3, 4)$, so $R(3, 4) = 10$.

$17 \to (3, 3, 3), 16 \not\to (3, 3, 3)$, so $R(3, 3, 3) = 17$.

The $R(5, 5)$ is known to be at least 43 and this is conjectured to be the correct value. However, in order to prove that it is 43 (that is, for every graph G on 43 vertices either G or its complement contains a K_5), we would either have to be quite clever or perform a computer search. A (naive) computer search would involve examining each of the $2^{\binom{43}{2}} = 2^{902}$ 2-colorings of the complete graph on 43 vertices. 2^{903} is rather large (larger than the number of particles in the known universe), so a computer search is unfeasible.

Exercise 7.10. Prove that Ramsey's theorem implies the weak form of Szekeres' theorem (i.e., it gives a "Happy Ending").

Theorem 7.11 (Erdős-Szekeres, 1935). $R(k, \ell) \le \binom{k+\ell-2}{k-1}$, *i.e.,* $\binom{k+\ell}{k} \to (k + 1, \ell + 1)$.

From the above statement, it follows that $R(k, k) < 4^k$. Using the probabilistic method, Erdős proved that $2^{k/2} < R(k, k)$.

More information about Ramsey numbers can be found in [**?**], the dynamic survey of Stanislaw Radziszowski at `http://www.combinatorics.org/` and at web-pages of Stanislaw Radziszowski (`http://www.cs.rit.edu/~spr/homepage.html`) and Brendan McKay (`http://cs.anu.edu.au/~bdm/`).

[CG98] Fan Chung, Ron Graham: *Erdős on graphs. His legacy of unsolved problems.* A. K. Peters, Ltd., Wellesley, MA, 1998.