Discrete Math, 9th day, Wednesday 7/7/04
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1 Extremal Set Theory

Notation: $[n] = \{1, \ldots, n\}$. The **incidence vector** of a set $A \subseteq [n]$ is defined as $\nu_A \in \mathbb{F}_2^n$ where $(\nu_A)_i = 1$ if and only if $i \in A$. **Standard inner product** of two vectors $a, b \in \mathbb{F}^n$ is defined as $a \cdot b = \sum_{i=1}^n a_i b_i$. Notice that $\nu_A \cdot \nu_B = |A \cap B|$ and $\nu_A \cdot \nu_A = |A|$.

Theorem 9.1 (Fisher's Inequality). *Let $A_1, \ldots, A_m \subseteq [n]$, and $a \geq 1$. If $|A_i \cap A_j| = a$ for every $i \neq j$, then $m \leq n$.*

In 1949 R.C. Bose proved Fisher's Inequality by showing linear independence of a well-chosen set of vectors over a well-chosen field, establishing the "linear algebra method" in combinatorics. The proof is based on the following lemma:

Lemma 9.2. *The incidence vectors of the A_i are linearly independent.*

If the size of one of the sets is a, e.g. $|A_i| = a$, then $A_j \supseteq A_i$ for every j and the system forms a **sunflower** (no two sets intersect outside the common intersection of all sets).

1.1 Eventown

Suppose there is a town of n citizens and there are m clubs $A_1, \ldots, A_m \subseteq [n]$. The lawmakers tend to create new laws and curiously examine the highest possible number of clubs under the current set of rules.

Rules in the Eventown:

(0) The clubs have to be distinct.

(1) Each club has even number of members.

(2) $|A_i \cap A_j|$ is even for every i, j.

Under rule (0), there could be total 2^n clubs. Under rules (0) and (1), the total number of clubs drops to 2^{n-1}. If all three rules need to be satisfied, the "married couples solution" provides $2^{n/2}$ clubs. This solution is **maximal**, i. e., the solution cannot be extended, adding another club would violate one of the rules. Is this solution also **maximum**, i. e., there is no solution with higher number of clubs?

Exercise 9.3. In Eventown, $m \leq 2^{\lfloor n/2 \rfloor}$. (The "married couples solution" is maximum.) HINT. Prove that the statement follows from Exercise **??**.

Exercise 9.4. Find another maximum solution which contains three clubs A_1, A_2, A_3 such that $|A_1 \cap A_2 \cap A_3|$ is odd. (So not all maximum solutions are isomorphic.)

Exercise 9.5. In Eventown every maximal set of clubs is maximum.

1.2 Oddtown

Rules in Oddtown:

(1) $|A_i|$ is odd for every i

(2) $|A_i \cap A_j|$ is even for every $i \neq j$

Exercise 9.6. The number of ways to create a system of n clubs in Oddtown is $\geq 2^{n^2/8}$.

Theorem 9.7 (Oddtown Theorem, Berlekamp). *In Oddtown, $m \leq n$.*

Lemma 9.8. *Under Oddtown rules the incidence vectors of the clubs are linearly independent.*

Exercise 9.9. Prove the previous lemma over the following fields: (a) over $\mathbb{Q}$, (b) over $\mathbb{F}_2$, (c) over $\mathbb{R}$.

Definition 9.10. For $A \subseteq [n]$, the **incidence vector** (or **characteristic vector**) of A is the vector $v_A = (\alpha_1, \ldots, \alpha_n)$ where

$$\alpha_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}.$$

Let B be a matrix with rows v_{A_i}. Part (a) implies that B has a full rank over $\mathbb{Q}$. Notice that rank is invariant under extension of the field (Gaussian elimination process keeps all coefficients in the original field), therefore part (c) is a consequence of part (a). Notice that $\mathbb{F}_2$ is not a subfield of $\mathbb{Q}$. However, part (a) follows from (b) and the following exercise.

Exercise 9.11. Let A be a $(0, 1)$-matrix, i. e., its entries are from $\{0, 1\}$. Let $\mathrm{rk}_p(A)$ denote the rank of A over $\mathbb{F}_p$ and let $\mathrm{rk}_0(A)$ be its rank over $\mathbb{Q}$. Prove: $\mathrm{rk}_p(A) \leq \mathrm{rk}_0(A)$.

Definition 9.12. Vectors $a, b \in \mathbb{F}^n$ are **perpendicular**, denoted $a \perp b$, if $a \cdot b = 0$. For $S \subseteq \mathbb{F}^n$, the set $S^\perp = \{x \mid (\forall y \in S)(x \perp y)\}$ is called **S-perp**. A vector $v \in \mathbb{F}^n$ is called **isotropic** if $v \perp v$. A set $\mathcal{U} \leq \mathbb{F}^n$ is **totally isotropic** if $\mathcal{U} \perp \mathcal{U}$.

Exercise 9.13. Prove $S^\perp = (\text{Span } S)^\perp$.

Exercise 9.14. Let $\mathcal{U} \leq \mathbb{F}^n$. Prove: $\dim(\mathcal{U}) + \dim(\mathcal{U}^\perp) = n$.

Let $\mathcal{U} \subseteq \mathbb{F}_2^n$ be a maximal set of clubs in Eventown. Then $\mathcal{U} \subseteq \mathcal{U}^\perp$ and therefore $\text{Span}(\mathcal{U}) \subseteq \text{Span}(\mathcal{U})^\perp$. Since $\mathcal{U}$ is maximal, we can conclude that $\mathcal{U} = \text{Span}(\mathcal{U})$. By previous exercise, $\dim(\mathcal{U}) \leq n/2$. Therefore $|\mathcal{U}| \leq |\mathbb{F}_2^{n/2}| = 2^{n/2}$, i. e., the married couples solution is optimal.

Exercise 9.15. For what p does there exist a nonzero isotropic vector in $\mathbb{F}_p^2$? (Answer is appealing.)

Exercise 9.16. Prove that there exists an n-dimensional totally isotropic subspace over $\mathbb{C}^{2n}$.

Question: How many sets can have pairwise intersection of size 0 or 1? If we take all sets of size at most 2, we get a set system of $\binom{n}{2} + n + 1$ sets.

Exercise$^+$ 9.17. Let $A_1, \ldots, A_m \subseteq [n]$ be such that $|A_i \cap A_j| \leq 1$ for $i \neq j$. Prove: $m \leq \binom{n}{2} + n + 1$
Hint. Use linear algebra method. The trick lies in finding a good set of vectors in dimension $\binom{n}{2} + n + 1$.