1 Extremal Set Theory

Notation: $[n] = \{1, \ldots, n\}$. The incidence vector of a set $A \subseteq [n]$ is defined as $\nu_A \in \mathbb{F}_2^n$ where $(\nu_A)_i = 1$ if and only if $i \in A$. Standard inner product of two vectors $a, b \in \mathbb{F}_2^n$ is defined as $a \cdot b = \sum_{i=1}^n a_i b_i$. Notice that $\nu_A \cdot \nu_B = |A \cap B|$ and $\nu_A \cdot \nu_A = |A|$.

**Theorem 9.1 (Fisher’s Inequality).** Let $A_1, \ldots, A_m \subseteq [n]$, and $a \geq 1$. If $|A_i \cap A_j| = a$ for every $i \neq j$, then $m \leq n$.

In 1949 R.C. Bose proved Fisher’s Inequality by showing linear independence of a well-chosen set of vectors over a well-chosen field, establishing the “linear algebra method” in combinatorics. The proof is based on the following lemma:

**Lemma 9.2.** The incidence vectors of the $A_i$ are linearly independent.

If the size of one of the sets is $a$, e.g. $|A_i| = a$, then $A_j \supseteq A_i$ for every $j$ and the system forms a **sunflower** (no two sets intersect outside the common intersection of all sets).

1.1 Eventown

Suppose there is a town of $n$ citizens and there are $m$ clubs $A_1, \ldots, A_m \subseteq [n]$. The lawmakers tend to create new laws and curiously examine the highest possible number of clubs under the current set of rules.

Rules in the Eventown:

(0) The clubs have to be distinct.

(1) Each club has even number of members.
(2) \(|A_i \cap A_j|\) is even for every \(i, j\).

Under rule (0), there could be total \(2^n\) clubs. Under rules (0) and (1), the total number of clubs drops to \(2^{n-1}\). If all three rules need to be satisfied, the “married couples solution” provides \(2^{n/2}\) clubs. This solution is maximal, i.e., the solution cannot be extended, adding another club would violate one of the rules. Is this solution also maximum, i.e., there is no solution with higher number of clubs?

**Exercise 9.3.** In Eventown, \(m \leq 2^{\lfloor n/2 \rfloor}\). (The “married couples solution” is maximum.) Hint. Prove that the statement follows from Exercise ??.

**Exercise 9.4.** Find another maximum solution which contains three clubs \(A_1, A_2, A_3\) such that \(|A_1 \cap A_2 \cap A_3|\) is odd. (So not all maximum solutions are isomorphic.)

**Exercise 9.5.** In Eventown every maximal set of clubs is maximum.

### 1.2 Oddtown

Rules in Oddtown:

1. \(|A_i|\) is odd for every \(i\)
2. \(|A_i \cap A_j|\) is even for every \(i \neq j\)

**Exercise 9.6.** The number of ways to create a system of \(n\) clubs in Oddtown is \(\geq 2^{n^2/8}\).

**Theorem 9.7 (Oddtown Theorem, Berlekamp).** In Oddtown, \(m \leq n\).

**Lemma 9.8.** Under Oddtown rules the incidence vectors of the clubs are linearly independent.

**Exercise 9.9.** Prove the previous lemma over the following fields: (a) over \(\mathbb{Q}\), (b) over \(\mathbb{F}_2\), (c) over \(\mathbb{R}\).

**Definition 9.10.** For \(A \subseteq [n]\), the incidence vector (or characteristic vector) of \(A\) is the vector \(v_A = (\alpha_1, \ldots, \alpha_n)\) where

\[
\alpha_i = \begin{cases} 
1 & \text{if } i \in A \\
0 & \text{if } i \notin A
\end{cases}
\]

Let \(B\) be a matrix with rows \(v_{A_i}\). Part (a) implies that \(B\) has a full rank over \(\mathbb{Q}\). Notice that rank is invariant under extension of the field (Gaussian elimination process keeps all coefficients in the original field), therefore part (c) is a consequence of part (a). Notice that \(\mathbb{F}_2\) is not a subfield of \(\mathbb{Q}\). However, part (a) follows from (b) and the following exercise.

**Exercise 9.11.** Let \(A\) be a \((0,1)\)-matrix, i.e., its entries are from \(\{0,1\}\). Let \(\text{rk}_p(A)\) denote the rank of \(A\) over \(\mathbb{F}_p\) and let \(\text{rk}_0(A)\) be its rank over \(\mathbb{Q}\). Prove: \(\text{rk}_p(A) \leq \text{rk}_0(A)\).
Definition 9.12. Vectors $a, b \in \mathbb{F}^n$ are perpendicular, denoted $a \perp b$, if $a \cdot b = 0$. For $S \subseteq \mathbb{F}^n$, the set $S^\perp = \{x \mid (\forall y \in S)(x \perp y)\}$ is called $S$-perp. A vector $v \in \mathbb{F}^n$ is called isotropic if $v \perp v$. A set $\mathcal{U} \subseteq \mathbb{F}^n$ is totally isotropic if $\mathcal{U} \perp \mathcal{U}$.

Exercise 9.13. Prove $S^\perp = (\text{Span } S)^\perp$.

Exercise 9.14. Let $\mathcal{U} \subseteq \mathbb{F}^n$. Prove: $\dim(\mathcal{U}) + \dim(\mathcal{U}^\perp) = n$.

Let $\mathcal{U} \subseteq \mathbb{F}_2^n$ be a maximal set of clubs in Eventown. Then $\mathcal{U} \subseteq \mathcal{U}^\perp$ and therefore $\text{Span}(\mathcal{U}) \subseteq \text{Span}(\mathcal{U})^\perp$. Since $\mathcal{U}$ is maximal, we can conclude that $\mathcal{U} = \text{Span}(\mathcal{U})$. By previous exercise, $\dim(\mathcal{U}) \leq n/2$. Therefore $|\mathcal{U}| \leq |\mathbb{F}_2^{n/2}| = 2^{n/2}$, i.e., the married couples solution is optimal.

Exercise 9.15. For what $p$ does there exist a nonzero isotropic vector in $\mathbb{F}_p^2$? (Answer is appealing.)

Exercise 9.16. Prove that there exists an $n$-dimensional totally isotropic subspace over $\mathbb{C}^{2n}$.

Question: How many sets can have pairwise intersection of size 0 or 1? If we take all sets of size at most 2, we get a set system of $\binom{n}{2} + n + 1$ sets.

Exercise$^+$ 9.17. Let $A_1, \ldots, A_m \subseteq [n]$ be such that $|A_i \cap A_j| \leq 1$ for $i \neq j$. Prove: $m \leq \binom{n}{2} + n + 1$

Hint. Use linear algebra method. The trick lies in finding a good set of vectors in dimension $\binom{n}{2} + n + 1$. 

3