

Discrete Math, 11th day, Tuesday 7/13/04
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1 Totally unimodular matrices

Definition 11.1. A matrix is **totally unimodular** if the determinant of every square submatrix is 0, 1, or -1 .

Definition 11.2. Let G be a digraph with n vertices and m edges. The **incidence matrix** A of G is a $n \times m$ matrix with rows corresponding to vertices and columns corresponding to edges s.t.

$$A[v, e] = \begin{cases} 0 & \text{if } v \text{ is not incident to } e, \\ 1 & \text{if } e \text{ starts at } v, \\ -1 & \text{if } e \text{ ends at } v. \end{cases}$$

Exercise 11.3. Prove: The incidence matrix of a digraph is totally unimodular.

2 Latin squares

Definition 11.4. Let S be an n -element set of symbols. A $n \times n$ matrix with entries from S is called a **Latin square** if each symbol appears exactly once in each row and in each column. $L(n)$ denotes the number of $n \times n$ Latin squares.

Theorem 11.5. For every $\varepsilon > 0$ there exists n_0 such that for every $n > n_0$

$$n^{(1-\varepsilon)n^2} \leq L(n) \leq n^{n^2}$$

Rephrasing the theorem we get $\log L(n) \sim n^2 \log n$.

We say that two Latin squares A and B are **equivalent**, denoted $A \sim B$, if A is obtained from B by a sequence of the following: (1) permuting rows, (2) permuting columns, (3) permuting symbols, (4) permuting the roles of rows, column indices, and symbols.

An example of type (4) equivalence would replace entry a in row b , column c by entry c in row a , column b .

Exercise 11.6. The number of Latin squares equivalent to a given Latin square is $\leq 6(n!)^3$.

Exercise 11.7. Let $\tilde{L}(n)$ be the number of inequivalent Latin squares of order n . Prove $\log \tilde{L}(n) \sim \log L(n)$.

It is easy to prove the upper bound from Theorem 11.5, since $L(n) < (n!)^n < n^{n^2}$ (why?). The lower bound is related to counting perfect matchings in a bipartite graph.

3 Counting perfect matchings in a bipartite graph; the permanent

A graph G is **bipartite** if its vertex set can be partitioned into V_1 and V_2 so that every edge in G has one endpoint in V_1 and the other in V_2 . A **matching** $M \subseteq E(G)$ is a set of edges such that no two edges in M share an endpoint. A matching is **perfect** if it contains $|V(G)|/2$ edges.

Definition 11.8. Let $A = (\alpha_{i,j})_{n \times n}$ be a matrix. The **permanent** of A is defined as

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^n \alpha_{i, \sigma(i)},$$

where S_n is the set of all permutations of $[n] = \{1, \dots, n\}$.

Let G be a bipartite graph with partitions V_1 and V_2 where $V_1 = [k_1] \times \{1\}$ and $V_2 = [k_2] \times \{2\}$. The **incidence matrix** of G is a $k_1 \times k_2$ matrix $M = (\alpha_{i,j})$ defined by

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } (i, 1) \sim (j, 2), \text{ i. e., } (i, 1) \text{ is adjacent to } (j, 2), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 11.9. Let G be a bipartite graph with $|V_1| = |V_2|$ and let M be its incidence matrix. Then the number of perfect matchings of G is $\text{per}(M)$.

Theorem 11.10. A regular bipartite graph of degree r with $n + n$ vertices has $> (r/e)^n$ perfect matchings.

4 Doubly Stochastic Matrices

Definition 11.11. A $n \times n$ matrix is called **stochastic** if all its entries are nonnegative and every row sums to 1. Matrix A is **doubly stochastic** if both A and A^T are stochastic, i. e., the entries are nonnegative and every row and every column sums to 1.

Let I be the identity matrix and J be the all-ones matrix.

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

Example 11.12. I and $\frac{1}{n}J$ are doubly stochastic.

Exercise 11.13. $n! > (n/e)^n$. (HINT: $e^x = \sum_{n=0}^{\infty} x^n/n!$)

Clearly, $\text{per}(I) = 1$ and $\text{per}(\frac{1}{n}J) = \frac{n!}{n^n} > \frac{1}{e^n}$.

Comment on how not to prove this inequality: Notice that the **Stirling's formula**

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

implies the inequality $n! > (\frac{n}{e})^n$ **only** for sufficiently large n , while the above method proves it for **every** n . A more precise Stirling's formula

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{\vartheta_n}{12n}\right),$$

where $|\vartheta_n| \leq 1$, would imply the inequality for every n .

Exercise 11.14. If A is doubly stochastic then $\text{per}(A) \leq 1$. The equality holds exactly when A is a permutation matrix (i. e., A has exactly one 1 in each row and column, otherwise zeros.)

Theorem 11.15 (The Permanent Inequality). *If A is doubly stochastic then*

$$\text{per } A \geq \text{per} \left(\frac{1}{n}J\right) = \frac{n!}{n^n}.$$

The theorem was conjectured by van der Waerden and was known as van der Waerden's Permanent Conjecture. It was proved independently by Egorichev and Falikman in 1980.

Exercise 11.16. Let M be the incidence matrix of a regular bipartite graph of degree r . Prove that $\frac{1}{r}M$ is doubly stochastic.

We show that the permanent inequality implies Theorem ???. Notice that $\text{per}(\alpha A) = \alpha^n \text{per}(A)$. Let M be the incidence matrix of the bipartite graph from Theorem ??. By applying the permanent inequality and Exercise ??? we get

$$\text{per}(M) = r^n \text{per} \left(\frac{1}{r}M\right) > r^n \frac{1}{e^n}.$$

5 Back to Latin squares

Definition 11.17. For $k \leq n$, a $k \times n$ **Latin rectangle** is a $k \times n$ matrix of n symbols such that every symbol appears **at most** once in each row and in each column. $L(k, n)$ denotes the number of $k \times n$ Latin rectangles.

Obviously, $L(n, k) \leq (n!)^k$ and $L(n, 1) = n!$.

Exercise 11.18. Prove: $L(n, 2) \sim \frac{(n!)^2}{e}$.

Theorem 11.19. Let $k < n$. Every $k \times n$ Latin rectangle can be extended to a $(k + 1) \times n$ Latin rectangle.

Theorem 11.20. Every regular bipartite graph of degree ≥ 1 has a perfect matching.

Exercise 11.21. Prove Theorem ?? using Theorem ??.

Suppose we have $k \times n$ Latin rectangle L . Define a bipartite graph G on $n + n$ vertices as follows. An edge goes from $(i, 1)$ to $(j, 2)$ if the i -th column in L does not contain the symbol j . This way we obtain a regular bipartite graph (of degree $n - k$). The perfect matchings of G correspond to all possible extensions of L to a $(k + 1) \times n$ Latin rectangle.

Therefore $L(n, k + 1) \geq L(n, k) \left(\frac{n-k}{e}\right)^n$, implying

$$L(n) = L(n, n) \geq \left(\frac{n}{e}\right)^n \left(\frac{n-1}{e}\right)^n \left(\frac{n-2}{e}\right)^n \cdots \left(\frac{1}{e}\right)^n = \frac{(n!)^n}{e^{n^2}}.$$

Exercise 11.22. Prove: $n \ln(n!) - n^2 \sim n \ln(n!)$; therefore $\log L(n) \sim n^2 \log n$.

6 Orthogonal Latin Squares

Definition 11.23. Let A and B be two $n \times n$ Latin squares. A and B are **orthogonal** if the set $\{(a_{i,j}, b_{i,j}) \mid i, j \in [n]\}$ contains all the n^2 distinct pairs, i. e., no pair $(a_{i,j}, b_{i,j})$ is repeated.

Orthogonal Latin squares are related to **Euler's "36 officers" problem**: There are 36 officers, 6 divisions and 6 ranks. The task is to assign a division and a rank to every officer so that no two officers are assigned the same combination of division and rank; and arrange the officers in 6×6 array such that each division is represented in each row and each column and each rank is represented in each row and each column.

Exercise 11.24. Let p be a prime. Prove that there exist $p - 1$ pairwise orthogonal Latin squares of order p .

Exercise 11.25. Prove previous exercise for p a prime power.

Theorem 11.26 (Tarry'1900). *Euler's 36 officers problem does not have a solution, i. e., there does not exist a pair of orthogonal 6×6 Latin squares. (Proof: tedious.)*

Exercise 11.27. Prove that for every n there are at most $n - 1$ pairwise orthogonal $n \times n$ Latin squares.

Exercise 11.28. Show that the following are equivalent: (a) there exists a set of $n - 1$ pairwise orthogonal $n \times n$ Latin squares, (b) there exists a projective plane of order n .

Exercise 11.29. Prove: $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$.

Exercise 11.30. Prove: $\binom{n}{k} < \left(\frac{en}{k}\right)^k$. Find an elegant solution, like the proof of $n! > \left(\frac{n}{e}\right)^n$.

Exercise 11.31. Prove: $\left(\frac{en}{k}\right)^k > \binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{0}$.

Exercise 11.32. In \mathbb{R}^n find cn^2 points with only two pairwise distances.

The vertices of a regular pentagon are an example of a set in \mathbb{R}^2 with only two pairwise distances.

Exercise 11.33. Prove: Any six points in the plane span at least three different distances.