1 Constructive Proofs of Negative Results in Ramsey Theory

Recall that we discussed the following results:

- \(4^n \to (n + 1, n + 1)\), proved by Erdős-Szekeres,
- \(2^{n/2} \not\to (n + 1, n + 1)\), showed by Erdős using a probabilistic proof of existence.

So far we have seen only one constructive proof of a negative result, the trivial example observed by Turán showing that \(n^2 \not\to (n + 1, n + 1)\).

**Theorem 12.1 (Zsigmond Nagy, 1973).** \(\binom{n}{3} \not\to (n + 1, n + 1)\) can be proved constructively.

Nagy’s construction defines the graph as follows. The vertices are all the 3-element subsets of \([n]\). Sets \(A\) and \(B\) are adjacent if \(|A \cap B| = 1\). The Erdős-deBruijn Theorem, a special case of Fisher’s inequality for sets with intersection size 1, implies that Nagy’s graph does not contain a clique of size \(n + 1\). By the Oddtown Theorem the graph does not contain an ant clique of size \(n + 1\).

The best known explicit construction is by Frankl and Wilson.

**Theorem 12.2 (Frankl - Wilson).** Let \(\varepsilon > 0\). For sufficiently large \(n\) one can construct a graph of at most \(n^{(1-\varepsilon)\frac{\ln n}{\ln \ln n}}\) vertices with no clique or ant clique of size \(n + 1\).

Let \(p\) be a prime. The vertices in the Frankl - Wilson graph are all the subsets of \([2p^2 - 1]\) of size \(p^2 - 1\). Sets \(A\) and \(B\) are adjacent if \(|A \cap B| \equiv -1 \pmod{p}\).

**Claim 12.3.** The Frankl - Wilson example proves that

\[
\binom{2p^2 - 1}{p^2 - 1} \not\to \binom{2p^2 - 1}{p - 1} + 1.
\]
The proof of the claim is based on two theorems on extremal set theory:

**Theorem 12.4 (Ray-Chaudhuri - Wilson, 1975).** Fix \( k \) and let \( l_1 < \cdots < l_s < k \). If \( A_1, \ldots, A_m \subseteq [n] \) are sets of size \( k \) such that \( |A_i \cap A_j| \in \{l_1, \ldots, l_s\} \) for every \( i \neq j \), then \( m \leq \binom{n}{s} \).

**Exercise 12.5.** Prove that the Ray-Chaudhuri - Wilson Theorem is tight, i.e. find \( \binom{n}{s} \) sets with \( s \) different intersection sizes.

Frankl and Wilson generalized the Ray-Chaudhuri - Wilson Theorem:

**Theorem 12.6 (Frankl - Wilson, 1981).** Let \( p \) be a prime and let \( k, l_1, \ldots, l_s \in \mathbb{Z}_p \) be such that \( k \not\equiv l_1, \ldots, l_s \pmod{p} \). If \( A_1, \ldots, A_m \subseteq [n] \) are sets of size \( k \) such that \( |A_i \cap A_j| \in \{l_1, \ldots, l_s\} \pmod{p} \) for every \( i \neq j \), then \( m \leq \binom{n}{s} \).

A clique in the Frankl - Wilson graph corresponds to a set system \( A_1, \ldots, A_m \), such that \( |A_i| = p^2 - 1 \) for every \( i \), such that \( |A_i \cap A_j| \in \{p - 1, 2p - 1, \ldots, p(p - 1) - 1\} \). Thus the Ray-Chaudhuri - Wilson Theorem implies that \( m \leq \binom{2p^2 - 1}{p - 1} \).

An anticlique corresponds to a set system \( B_1, \ldots, B_m \subseteq [2p^2 - 1] \), \( |B_i| = p^2 - 1 \equiv -1 \pmod{p} \) for every \( i \), such that \( |B_i \cap B_j| \in \{0, 1, \ldots, p - 2\} \pmod{p} \). By the Frankl - Wilson Theorem, \( m \leq \binom{2p^2 - 1}{p - 1} \).

### 2 Bipartite Ramsey Theory

We define a bipartite version of the Erdős-Rado arrow.

**Definition 12.7.** We say that \( a \sim (b, c) \) if every bipartite graph \( G \) with \( a \) vertices contains a bipartite clique \( K_{b,b} \) or the complement \( \overline{G} \) contains a bipartite clique \( K_{c,c} \).

**Exercise 12.8.** Prove: \( 4^n \sim (n + 1, n + 1) \).

**Exercise 12.9.** Prove: \( 2^{n/2} \not\sim (n + 1, n + 1) \).

**Hint.** Probabilistic proof of existence.

### 3 Hadamard Matrices

**Theorem 12.10 (Hadamard's Inequality).** Let \( A \in M_n(\mathbb{R}) \), i.e. \( A \) is a \( n \times n \) matrix over \( \mathbb{R} \). Then

\[
|\det(A)| \leq \prod_{i=1}^{n} \|a_i\|,
\]

where \( a_i \) is the vector in the \( i \)-th row of \( A \) and \( \|a_i\| = \sqrt{\sum_{j=1}^{n} a_{i,j}^2} \) is its norm. The equality holds if and only if there exists a zero row or if the rows are pairwise orthogonal.
Definition 12.11. An Hadamard matrix is a $\pm 1$-matrix with all rows orthogonal.

The Sylvester matrices are $2^k \times 2^k$ matrices defined by the following matrix recurrence.

\[ H_0 = \begin{pmatrix} 1 \end{pmatrix} \]

\[ H_{k+1} = \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix} \quad \text{for } k > 0 \]


Definition 12.13. $n$ is an Hadamard number if there exists an $n \times n$ Hadamard matrix.

Exercise 12.14. If $n$ is an Hadamard number then $n \leq 2$ or $n$ is divisible by $4$.

Exercise† 12.15. If $p \equiv -1 \mod 4$ is a prime power, then $p + 1$ is an Hadamard number. Hint: Quadratic residues.

Exercise 12.16. If $H$ is an Hadamard matrix then $H^T$ is also an Hadamard matrix. Hint: Examine $HH^T$.

Exercise 12.17. Construct a matrix with all rows orthogonal such that the columns are not orthogonal.

Definition 12.18. Let $M = (m_{i,j})$ be a $n \times n$ matrix and let $I,J \subseteq [n]$. A rectangle is a submatrix of $M$ corresponding to rows defined by $I$ and columns defined by $J$. The discrepancy of the rectangle is defined as

\[ \text{disc}_{I,J}(M) = \left| \sum_{i \in I, j \in J} m_{i,j} \right|. \]

Theorem 12.19 (Lindsay’s Inequality). Let $a = |I|$ and $b = |J|$. If $H$ is an Hadamard matrix then

\[ \text{disc}_{I,J}(H) \leq \sqrt{nab} \]

Note 12.20 (Back to bipartite Ramsey numbers.). Let $M$ be an incidence matrix of a bipartite graph, with zeros replaced by $-1$. The rectangle corresponding to a bipartite clique or anticlique of size $t$ has discrepancy $t^2$. By Lindsay’s Inequality, $t^2 \leq t \sqrt{n}$, so $t \leq \sqrt{n}$. Therefore

\[ n^2 \not\to (n + 1, n + 1). \]

The proof of Lindsay’s inequality is based on the following facts:

Theorem 12.21 (Cauchy-Schwarz Inequality). Let $a, b \in \mathbb{R}^n$. Then

\[ |a \cdot b| \leq \|a\| \|b\|, \]

where $\cdot$ denotes the standard inner product.
Definition 12.22. A matrix $K$ is orthogonal if $K^T = K^{-1}$.

Lemma 12.23. If a $n \times n$ matrix $K$ is orthogonal, then $\|Kx\| = \|x\|$ for every vector $x \in \mathbb{F}^n$.

Proof: [of Lindsay’s Inequality] Let $e_S \in \mathbb{R}^n$ be the characteristic vector of the set $S \subseteq [n]$. Then

$$\sum_{i \in I, j \in J} m_{i,j} = e_I^* Me_J$$

If $H$ is Hadamard then $\frac{1}{\sqrt{n}}H$ is orthogonal. By Cauchy-Schwarz and the above lemma,

$$\text{disc}_{I,J}(H) = \|e_I^* He_J\| \leq \|e_I\|\|He_J\| = \sqrt{a\sqrt{nb}}$$

Exercise 12.24. Prove: If $a$ and $b$ are Hadamard numbers then $ab$ is a Hadamard number.

Conjecture 12.25. If $n$ is divisible by 4 then $n$ is a Hadamard number.

Definition 12.26. The upper density of a set $A \subseteq \mathbb{N}$ is

$$\limsup_{n \to \infty} \frac{|A(n)|}{n},$$

where $A(n) = \{x \in A \mid x \leq n\}$. The lower density is

$$\liminf_{n \to \infty} \frac{|A(n)|}{n}.$$

We say that $A$ has density $\gamma$ if $\gamma$ is both the lower and the upper density.

Exercise 12.27. Construct a set with lower density 0 and upper density 1.

Exercise 12.28. The upper density of Hadamard numbers is $\leq 1/4$.

The Conjecture ?? would imply that the density of Hadamard numbers is 1/4.

OPEN PROBLEM: Is the (upper) density of Hadamard numbers positive? The Sylvester matrices example shows that there are infinitely many Hadamard matrices. The quadratic residue Hadamard matrices give asymptotic density $\frac{1}{2\ln n}$. However, the density of this set is still 0.