Discrete Math, 12th day, Thursday 7/15/04
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1 Constructive Proofs of Negative Results in Ramsey Theory

Recall that we discussed the following results:

- $4^n \to (n+1, n+1)$, proved by Erdős-Szekeres,
- $2^{n/2} \not\to (n+1, n+1)$, showed by Erdős using a probabilistic proof of existence.

So far we have seen only one **constructive** proof of a negative result, the trivial example observed by Turán showing that $n^2 \not\to (n+1, n+1)$.

Theorem 12.1 (Zsigmond Nagy, 1973). *$\binom{n}{3} \not\to (n+1, n+1)$ can be proved constructively.*

Nagy's construction defines the graph as follows. The vertices are all the 3-element subsets of $[n]$. Sets A and B are adjacent if $|A \cap B| = 1$. The Erdős-deBruijn Theorem, a special case of Fisher's inequality for sets with intersection size 1, implies that Nagy's graph does not contain a clique of size $n+1$. By the Oddtown Theorem the graph does not contain an anticlique of size $n+1$.

The best known explicit construction is by Frankl and Wilson.

Theorem 12.2 (Frankl - Wilson). *Let $\varepsilon > 0$. For sufficiently large n one can construct a graph of at most $n^{(1-\varepsilon)\frac{\ln n}{4 \ln \ln n}}$ vertices with no clique or anticlique of size $n+1$.*

Let p be a prime. The vertices in the Frankl - Wilson graph are all the subsets of $[2p^2 - 1]$ of size $p^2 - 1$. Sets A and B are adjacent if $|A \cap B| \equiv -1 \mod p$.

Claim 12.3. *The Frankl - Wilson example proves that*

$$\binom{2p^2-1}{p^2-1} \not\to \binom{2p^2-1}{p-1} + 1.$$

The proof of the claim is based on two theorems on extremal set theory:

Theorem 12.4 (Ray-Chaudhuri - Wilson, 1975). *Fix k and let $l_1 < \cdots < l_s < k$. If $A_1, \ldots, A_m \subseteq [n]$ are sets of size k such that $|A_i \cap A_j| \in \{l_1, \ldots, l_s\}$ for every $i \neq j$, then $m \leq \binom{n}{s}$.*

Exercise 12.5. Prove that the Ray-Chaudhuri - Wilson Theorem is tight, i.e. find $\binom{n}{s}$ sets with s different intersection sizes.

Frankl and Wilson generalized the Ray-Chaudhuri - Wilson Theorem:

Theorem 12.6 (Frankl - Wilson, 1981). *Let p be a prime and let $k, l_1, \ldots, l_s \in \mathbb{Z}_p$ be such that $k \not\equiv l_1, \ldots, l_s \mod p$. If $A_1, \ldots, A_m \in [n]$ are sets of size k such that $|A_i \cap A_j| \in \{l_1, \ldots, l_s\} \mod p$ for every $i \neq j$, then $m \leq \binom{n}{s}$.*

A clique in the Frankl - Wilson graph corresponds to a set system $A_1, \ldots, A_m$, $|A_i| = p^2 - 1$ for every i, such that $|A_i \cap A_j| \in \{p-1, 2p-1, \ldots, p(p-1)-1\}$. Thus the Ray-Chaudhuri - Wilson Theorem implies that $m \leq \binom{2p^2-1}{p-1}$.

An anticlique corresponds to a set system $B_1, \ldots, B_m \subseteq [2p^2-1]$, $|B_i| = p^2 - 1 \equiv -1 \mod p$ for every i, such that $|B_i \cap B_j| \in \{0, 1, \ldots, p-2\} \mod p$. By the Frankl - Wilson Theorem, $m \leq \binom{2p^2-1}{p-1}$.

2 Bipartite Ramsey Theory

We define a bipartite version of the Erdős-Rado arrow.

Definition 12.7. We say that $a \rightsquigarrow (b, c)$ if every bipartite graph G with a vertices contains a bipartite clique $K_{b,b}$ or the complement $\overline{G}$ contains a bipartite clique $K_{c,c}$.

Exercise 12.8. Prove: $4^n \rightsquigarrow (n+1, n+1)$.

Exercise 12.9. Prove: $2^{n/2} \not\rightsquigarrow (n+1, n+1)$.
Hint. Probabilistic proof of existence.

3 Hadamard Matrices

Theorem 12.10 (Hadamard's Inequality). *Let $A \in M_n(\mathbb{R})$, i.e. A is a $n \times n$ matrix over $\mathbb{R}$. Then*

$$|\det(A)| \leq \prod_{i=1}^{n} \|\mathbf{a}_i\|,$$

where $\mathbf{a}_i$ is the vector in the i-th row of A and $\|\mathbf{a}_i\| = \sqrt{\sum_{j=1}^{n} a_{i,j}^2}$ is its norm. The equality holds if and only if there exists a zero row or if the rows are pairwise orthogonal.

Definition 12.11. An **Hadamard matrix** is a ± 1-matrix with all rows orthogonal.

The **Sylvester matrices** are $2^k \times 2^k$ matrices defined by the following matrix recurrence.

$$\begin{array}{rcl} H_0 & = & (1) \\ H_{k+1} & = & \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix} \quad \text{for } k > 0 \end{array}$$

Exercise 12.12. Prove: H_k is an Hadamard matrix.

Definition 12.13. n is an **Hadamard number** if there exists a $n \times n$ Hadamard matrix.

Exercise 12.14. If n is an Hadamard number then $n \leq 2$ or n is divisible by 4.

Exercise$^+$ 12.15. If $p \equiv -1 \mod 4$ is a prime power, then $p+1$ is an Hadamard number.
Hint. Quadratic residues.

Exercise 12.16. If H is an Hadamard matrix then H^T is also an Hadamard matrix.
Hint. Examine HH^T.

Exercise 12.17. Construct a matrix with all rows orthogonal such that the columns are not orthogonal.

Definition 12.18. Let $M = (m_{i,j})$ be a $n \times n$ matrix and let $I, J \subseteq [n]$. A **rectangle** is a submatrix of M corresponding to rows defined by I and columns defined by J. The **discrepancy** of the rectangle is defined as

$$\text{disc}_{I,J}(M) = \left| \sum_{i \in I, j \in J} m_{i,j} \right|.$$

Theorem 12.19 (Lindsay's Inequality). *Let $a = |I|$ and $b = |J|$. If H is an Hadamard matrix then*

$$\text{disc}_{I,J}(H) \leq \sqrt{nab}$$

Note 12.20 (Back to bipartite Ramsey numbers.). Let M be an incidence matrix of a bipartite graph, with zeros replaced by -1. The rectangle corresponding to a bipartite clique or anticlique of size t has discrepancy t^2. By Lindsay's Inequality, $t^2 \leq t\sqrt{n}$, so $t \leq \sqrt{n}$. Therefore

$$n^2 \not\to (n+1, n+1).$$

The proof of Lindsay's inequality is based on the following facts:

Theorem 12.21 (Cauchy-Schwarz Inequality). *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then*

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|,$$

where $\cdot$ denotes the standard inner product.

Definition 12.22. A matrix K is **orthogonal** if $K^T = K^{-1}$.

Lemma 12.23. *If a $n \times n$ matrix K is orthogonal, then $\|K\mathbf{x}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x} \in \mathbb{F}^n$.*

Proof: [of Lindsay's Inequality] Let $\mathbf{e}_S \in \mathbb{R}^n$ be the characteristic vector of the set $S \subseteq [n]$. Then

$$\sum_{i \in I, j \in J} m_{i,j} = \mathbf{e}_I^* M \mathbf{e}_J$$

If H is Hadamard then $\frac{1}{\sqrt{n}} H$ is orthogonal. By Cauchy-Schwarz and the above lemma,

$$\operatorname{disc}_{I,J}(H) = |e_I^* H e_J| \leq \|e_I\| \|H e_J\| = \sqrt{a}\sqrt{nb}$$

Exercise 12.24. Prove: If a and b are Hadamard numbers then ab is a Hadamard number.

Conjecture 12.25. *If n is divisible by 4 then n is a Hadamard number.*

Definition 12.26. The **upper density** of a set $A \subseteq \mathbb{N}$ is

$$\limsup_{n \to \infty} |A(n)/n|,$$

where $A(n) = \{x \in A \mid x \leq n\}$. The **lower density** is

$$\liminf_{n \to \infty} |A(n)/n|.$$

We say that A has density γ if γ is both the lower and the upper density.

Exercise 12.27. Construct a set with lower density 0 and upper density 1.

Exercise 12.28. The upper density of Hadamard numbers is $\leq 1/4$.

The Conjecture **??** would imply that the density of Hadamard numbers is $1/4$.

OPEN PROBLEM: Is the (upper) density of Hadamard numbers positive? The Sylvester matrices example shows that there are infinitely many Hadamard matrices. The quadratic residue Hadamard matrices give asymptotic density $\frac{1}{2 \ln n}$. However, the density of this set is still 0.