Discrete Math, 13th day, Tuesday 7/20/04
REU 2004. Info:
http://people.cs.uchicago.edu/∼laci/reu04.

Instructor: László Babai
Scribe: Charilaos Skiadas

1 The Gale-Berlekamp Switching game

In this game there are 2 players and they play on a $n \times n$ checkerboard. There are exactly two moves in this game. In the first move, player 1 assigns a sign (± 1) to every square. Then, the second player switches the signs in any rows and columns he wants. After he finishes, they sum all numbers in the final configuration and look at the absolute value of the result. This is the amount the 1st player pays to the 2nd player. So naturally, the 1st player wants to minimize this number, while the 2nd player wants to maximize it. We want to know what the value of this game is, at least asymptotically in n. The value is the amount that the player 2 is guaranteed to receive, or equivalently the largest amount that the player 1 will have to pay, assuming both players follow optimal strategies. This value is certainly is no more than n^2. Player 2 can easily achieve n in the following way: Make everything in the first row a plus, by switching columns. Then, if the sum is a, we could switch the first row to get sum $a - 2n$. Since $\max\{|a|, |a - 2n|\} \geq n$, we see that the value of the game is at least n. The question is: Can he achieve more?

Let us first make some observations: First of all, the order in which player 2 makes the switches does not matter, so we can assume that he switches the columns first, and then switches the rows. After he has switched the columns, then his best strategy is to switch the rows with negative sum, so that all the rows will have nonnegative sum. In particular, his row-switching moves are completely determined (except for the rows with zero sum, where switching makes no difference). If player 2 could arrange it so that many row sums are large, then he could improve his win.

So our goal should be to switch the columns so as to maximize the absolute value of the sum on an "average" row. Our goal is to achieve approximately $c\sqrt{n}$ gain per row. So this would give $c'n^{3/2}$ total gain. The answer is to randomly decide whether to switch a column or not. The main observation is that a random sequence of n pluses and minuses has expected absolute sum $\Theta(\sqrt{n})$:

Assume for simplicity that n is even. We see that the probability

$$P(n \text{ coin flips, } k \text{ heads}) = \frac{\binom{n}{k}}{2^n}.$$

Asymptotically,

$$\frac{\binom{n}{\frac{n}{2}}}{2^n} = \frac{\frac{n!}{(\frac{n}{2})!^2}}{2^n} = \ldots \sim \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}} = \frac{c}{\sqrt{n}},$$

were we used Stirling's formula to estimate the factorials. So

$$P(\#\text{heads within } \frac{n}{2} \pm k) = \frac{\binom{n}{\frac{n}{2}-k} + \cdots + \binom{n}{\frac{n}{2}+k}}{2^n}.$$

There are precise estimates for this sum, but for our purposes, a rough estimate is enough. Since the middle binomial coefficient is the largest one, we see that the above sum is less than

$$\frac{(2k+1)\binom{n}{\frac{n}{2}}}{2^n} \sim (2k+1)\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}}.$$

As long as $(2k+1)\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}} \le \frac{1}{2}$, i. e., as long as $k < c\sqrt{n}$, where $c := \frac{\sqrt{\pi}}{4\sqrt{2}}$, the above probability is less than $\frac{1}{2}$. Then $P(\#\text{heads is within } \frac{n}{2} \pm c\sqrt{n})$ is asymptotically at most $\frac{1}{2}$.

(Note: The Central Limit Theorem, says that asymptotically, the binomial coefficients fit on a bell curve, and can hence obtain a better value for the constant above.)

So if in our game the 2nd player randomly switches the columns, then we expect at least roughly half of the rows to have absolute sum greater than $c\sqrt{n}$.

This means that on average, at least $\frac{n}{2}$ of the rows have this advantage. But this means that it is possible for player 2 to achieve that advantage, which would give him overall at least $\frac{1}{2}cn^{3/2}$ total gain.

The question now arises: Can we recommend a strategy for player 1 to force the total to be no more than that order? One could argue in a similar way with a random choice, but in this case there actually is a deterministic strategy. The idea is to use Hadamard matrices. Recall that a Hadamard matrix is a matrix with entries only ± 1, where the rows are orthogonal. As a consequence, all columns are orthogonal. We know that such matrices exist when $n = 2^k$ and when $n = p + 1$, where p is a prime and $p \equiv -1 \mod 4$. If H is a Hadamard matrix, then H provides a good strategy for player 1: Any switch of rows or columns gives again a Hadamard matrix. Then Lindsay's lemma tells us that a $a \times b$ submatrix has sum of entries less than or equal to $\sqrt{nab}$ in absolute value. For $a = b = n$, we get that the total sum of the entries in a Hadamard matrix is no more than $n^{3/2}$. Hence, in this case player 2 can't achieve a total gain of more than $n^{3/2}$.

If n is not itself a Hadamard number, just take a Hadamard number ℓ larger than it, no bigger than $2n$. (Doable, since between n and $2n$ there is always a power of 2.) Then take a Hadamard matrix of size $\ell \times \ell$, and choose your matrix to be any $n \times n$ submatrix of that. Then any row and column changes that player 2 performs, we can assume that they are performed to the whole matrix. By Lindsay's lemma we then get that the sum is no more than $\sqrt{\ell n^2} < \sqrt{2} n^{3/2}$. (Note: If we use Hadamard matrices of size $p+1$, we can actually get $1+\varepsilon$ as the constant instead of $\sqrt{2}$.) Notice that there is a gap between the constants we have obtained in the deterministic upper bound and the probabilistic lower bound for the value. While this gap can be reduced, the instructor believes it has not been closed: the lower bound differs from the upper bound by $cn^{3/2}$.

2 The RAY-CHAUDHURI - WILSON theorem

See separate handout.