1 Points in general position

Exercise 14.1. Find a curve \( f : \mathbb{R} \to \mathbb{R}^n \) mapping any \( n \) distinct points into \( n \) points in “general position”. In other words, if \( t_1, \ldots, t_n \in \mathbb{R} \) are distinct, then \( f(t_1), \ldots, f(t_n) \) are linearly independent.

Recall the definition of the Vandermonde determinant and the corresponding closed form expression

\[
V_n(t_1, \ldots, t_n) := \begin{vmatrix}
1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\
1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
1 & t_n & t_n^2 & \cdots & t_n^{n-1}
\end{vmatrix} = \prod_{i \neq j}(t_i - t_j)
\]

Then a solution to the above exercise is given by \( f(t) = (f_0(t), \ldots, f_{n-1}(t)) = (1, t, t^2, \ldots, t^{n-1}) \).

2 Pattern in proofs of inequalities using the Linear Algebra Method

The pattern can be summarized as follows. We have objects \( a_1, \ldots, a_m \in \Omega \), where \( \Omega \) is some abstract domain. We want to obtain an upper-bound on \( m \). We define functions \( f_1, \ldots, f_m : \Omega \to W \), where \( W \) is a vector space, satisfying the “diagonal condition”

\[
f_i(a_j) = \begin{cases} 
\neq 0 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]
The set of functions $\Omega \to W$ forms a vector space in $W^{\Omega}$. The diagonal condition implies linear independence of $f_1, \ldots, f_m$. If in addition we find another set of functions $g_1, \ldots, g_k$ such that $f_1, \ldots, f_m \in \text{Span}(g_1, \ldots, g_k)$ then $m \leq k$ follows by the Fundamental Fact of Linear Algebra. A weaker condition suffices for this.

**Claim 14.2.** Suppose the $f_i$ satisfy “triangular condition”:

$$f_i(a_j) = \begin{cases} 
\neq 0 & \text{if } i = j \\
0 & \text{if } i > j 
\end{cases}$$

Then the $f_i$ are linearly independent. (Note that we know nothing about $f_i(a_j)$ when $i < j$.)

**Exercise 14.3.** Prove Claim **14.2**.

**Claim 14.4.** If $W = \mathbb{F}$ and the matrix $(f_i(a_j))_{i,j=1}^m$ is nonsingular then the $f_i$ are linearly independent.

**Exercise 14.5.** Prove Claim **14.4**.

**Exercise 14.6.** Assuming $W = \mathbb{F}$, prove that the following statement is false: If the $f_i$ are linearly independent then the matrix $(f_i(a_j))_{i,j=1}^m$ is nonsingular.

The following theorem is a nonuniform version of the Ray-Chaudhuri – Wilson’s Theorem (the sets do not have to be equal size.)

**Theorem 14.7 (Frankl-Wilson).** Let $A_1, \ldots, A_m \subseteq [n]$ be such that $(\forall i \neq j)(|A_i \cap A_j| \in \{\ell_1, \ldots, \ell_s\})$. Then

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$$

The proof is a modification of the proof of the uniform case (all the $A_i$ are of the same size.) We cannot use the same functions $f_i(x)$ to prove the linear independence. The original $f_i$ were defined as

$$f_i(x) = \prod_{t=1}^s (v_i \cdot x - \ell_t),$$

where $x, v_i \in \mathbb{R}^n$, $v_i$ is the incidence vector of $A_i$. Notice that $f_i(v_i) \neq 0$ only if $|A_i| \not\in \{\ell_1, \ldots, \ell_s\}$. Therefore we define $g_i(x)$ as follows

$$g_i(x) = \prod_{t: i < |A_i|} (v_i \cdot x - \ell_t)$$

Then the $g_i$ satisfy

$$g_i(v_j) = \begin{cases} 
\neq 0 & \text{if } i = j \\
0 & \text{if } |A_i \cap A_j| < |A_i|, i.e. A_j \not\supseteq A_i 
\end{cases}$$

If we reorder the $A_i$ so that $A_j \not\supseteq A_i$ for $i > j$, then the condition on $g_i$ is a triangular condition. Therefore we could conclude that the $g_i$ are linearly independent. Such an ordering exists: it suffices to order the $A_i$ by their size — let $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. From this point on the proof follows along the lines of the uniform case.
3 Additional Exercises in Probability Theory

Exercise 14.8. A poker hand is a set of five cards. We say that the poker hand has a pair if there are two cards of the same kind (two Kings, or two 9s, for example). Compute the probability that a poker hand has at least one pair.

Definition 14.9. Two random variables are uncorrelated if $E(XY) = E(X)E(Y)$.

Exercise 14.10. Let $c$ and $d$ be constants. Prove: If $X$ and $Y$ are uncorrelated then $X + c$ and $Y + d$ are also uncorrelated.

Exercise 14.11. Suppose $X_1, \ldots, X_k$ are non-zero pairwise uncorrelated random variables with $E(X_i) = 0$. Prove: $X_1, \ldots, X_k$ are linearly independent (over $\mathbb{R}$).

Exercise 14.12. If $X_1, \ldots, X_m$ are random variables with $E(X_i) = 0$ then there exist pairwise uncorrelated random variables $Y_1, \ldots, Y_k$ with $E(Y_i) = 0$ such that $\text{Span}(X_1, \ldots, X_m) = \text{Span}(Y_1, \ldots, Y_k)$.

The process of finding $Y_1, \ldots, Y_k$ is called factor analysis in statistics. The $Y_i$ are called factors of the $X_i$.

Exercise 14.13. Prove: If there exist $k$ independent non-constant random variables over a probability space $\Omega$, then $|\Omega| \geq 2^k$.


(a) Prove: If $X_1, \ldots, X_k$ are non-constant pairwise independent random variables over $\Omega$, then $|\Omega| \geq k + 1$.

(b) Prove that this inequality is tight for $k = 2^\ell - 1$. 