Discrete Math, 15th day, Friday 7/23/04
REU 2004. Info:
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1 Bases for Vector Spaces of Polynomials

Suppose there are n cookies to distribute among k children. How many ways are there to distribute the cookies? First, consider the number of ways to distribute the cookies so that each child gets at least one cookie. If we suppose the i-th child gets x_i cookies, then we can solve the problem by looking for a solution to the equation $\sum_{i=1}^{k} x_i = n$ such that the unknowns are integers $x_i \geq 1$, $i \in \{1, \ldots, n\}$.

If the children are in order and we put the cookies in an order, all we have to do is insert partitions among n objects. The first set of cookies ends at the first inserted partition, and this set goes to the first child. The second set of cookies begins at the first inserted partition and ends at the second inserted partition, and this set goes to the second child, and so on. For n cookies, there are $n-1$ slots between cookies, and for k children, there are $k-1$ partitions that we must insert, so we get $\binom{n-1}{k-1}$ ways to distribute the cookies.

Suppose now that we want to allow the possibility that some children might not get a cookie. Then we are trying to solve $\sum_{i=1}^{k} y_i = n$ with $y_i \geq 0$. By giving one "dummy" cookie to each child before we distribute the cookies, we can reduce this to the previous problem. Instead of n cookies, however, we need $n+k$ cookies, so then we get $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ ways to distribute the cookies.

Now, suppose we look at homogeneous polynomials in r variables of degree d. They form a vector space with a basis of all monic monomials of degree d in the r variables $x_1, \ldots, x_r$. To compute the dimension of this space, we need to count those monomials.

Denote the number of such monomials by $H(r, k, \mathbb{F})$, where $\mathbb{F}$ is the field over which we are taking these polynomials. This number is the same as the number of solutions in the second cookie problem, but now we have degree d instead of n cookies, which we must divide between r variables instead of k children. Thus, we get $\binom{d+r-1}{r-1}$ monomials of degree d in r variables.

Now let $P(r, d, \mathbb{F})$ be the space of polynomials of degree $\leq d$ in $x_1, \ldots, x_r$. The condition here is that we are looking for $a_i \geq 0$ such that $\sum_{i=1}^{r} a_i \leq d$. We know how many ways we can do this for each $j \leq d$, so the total dimension is just the sum. To compute $\sum_{j=0}^{d} \binom{j+r-1}{r-1}$, we can use Pascal's triangle. In Pascal's triangle, the sum of two adjacent entries is the entry between them on the next line. If we look at the entries $\binom{j+r-1}{r-1}$ in Pascal's triangle, we can use the preceding fact to calculate $\sum_{j=0}^{d} \binom{j+r-1}{r-1} = \binom{r+d}{r}$.

As an alternative method, we can use a "dummy" child x_{r+1} to get the remaining cookies not distributed. That is, if we have $x_1^{a_1} \cdots x_r^{a_r}$ such that $\sum_{i=1}^{r} a_i \leq d$, the we can insert $x_{r+1}^{d-\sum_{i=1}^{r} a_i}$ to get a monomial in $r+1$ variables of degree d. Conversely, given a monomial of degree d in $r+1$ variables we can replace the $r+1$st variable by 1 to get a monomial of degree d in r variables. This directly gives $\binom{r+d}{r}$ as $P(r, d, \mathbb{F})$.

2 Projective Representation of a Graph

Definition 15.1. A **projective representation of a graph** G consists of the following information. Let W be a vector space over a field $\mathbb{F}$. To every vertex x in G, we assign a subspace $U(x) \leq W$ in such a manner, that $U(x) \cap U(y) \neq \{0\}$ iff $x \sim y$.

We want to minimize the dimension of the space W where a graph has such a representation.

Definition 15.2. The **projective dimension** of a graph G over a field $\mathbb{F}$ is

$$\mathrm{pdim}_{\mathbb{F}}(G) = \min\{\dim W : \text{ a projective representation of } G \text{ in } W \text{ exists}\}.$$

For example, for the empty graph $\bar{K}_n$, $\mathrm{pdim}_{\mathbb{F}}(\bar{K}_n) = 0$. For the complete graph K_n, $\mathrm{pdim}_{\mathbb{F}}(K_n) = 1$ by having each vertex correspond to the whole space.

For a cycle, say a 5-cycle, we produce a projective representation in $\mathbb{F}^4$. Assign nonzero vectors to each edge, and take the subspace corresponding to a vertex to be the subspace spanned by the vectors assigned to the incident edges. Since the subspaces associated to adjacent vertices will share a nonzero vector, the subspaces will have nonzero intersection. To prevent subspaces corresponding to nonadjacent vertices from having nonzero intersection, we need that any four of the vectors assigned to the edges be linearly independent, i.e., we need the five vectors $u_1, \ldots, u_5$ assigned to the edges to be in general position.

We know we can produce as many points in general position in our space as the number of elements in our field. So $\mathrm{pdim}_{\mathbb{F}}(C_n) \leq 4$ if $|\mathbb{F}| \geq n$.

Now look at a perfect matching graph with n vertices $\frac{n}{2}K_2$. If the field $\mathbb{F}$ has enough elements (at least $\frac{n}{2} - 1$), its projective dimension is 2: send each pair to a different line in $\mathbb{F}^2$.

Theorem 15.3. *Over any field, every graph has a projective representation*

Proof: Consider a vector space with basis a set of vectors in bijection with the set of edges on the graph. Then assign each vertex to the subspace spanned by the vectors corresponding to the incident edges. Actually we can do better: if $\Delta =$ maximum degree, then $\dim W = 2\Delta$ suffices because all we need is, from a vector space of that dimension, to pick the vectors so that they are in general position. This we can do as long as $|\mathbb{F}| \geq m$, where m is the number of edges.

So, in fact we have shown that:

Theorem 15.4. $\operatorname{pdim}_{\mathbb{F}} G \leq 2\Delta$.

In particular, for a bipartite graph, we get that its projective dimension is less than or equal to n.

Exercise 15.5. Improve this bound by a lot (at least to $o(n)$).

Exercise 15.6. Give a logarithmic upper bound on the complement of a perfect matching, i.e., show $\dim \overline{\left(\frac{n}{2}K_2\right)} \leq O(\log n)$.

Now, can we guarantee a lower bound on the projective dimension? (This question is related to sub-linear space complexity).

Given a machine with a big read-only table with n bits, and small read-write tables, give example of something that can't be computed in sub-linear space ($o(n)$).

The best lower bounds we have are $\log(n)$. We can prove though that almost all graphs have at least $\sqrt{n}$ lower bound.

OPEN PROBLEM(Concept introduced by Pudlák-Rödl): Construct an explicit family of graphs with projective dimension greater than n^c, where $c > 0$.

3 The Number of Zero-Patterns of a Sequence of Polynomials

See separate handout.