Discrete Math, 18th day, Wednesday 8/4/04
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1 Projective dimension of a graph, continued

Definition 18.1. A **projective representation** of a graph G over a field $\mathbb{F}$ is a vector space W over $\mathbb{F}$ and an assignment of subspaces to vertices $i \mapsto U_i \leq W$ such that $i \sim j$ (adjacent vertices) $\Leftrightarrow U_i \cap U_j \neq \{0\}$.

Projective dimension of a graph: $\mathrm{pdim}_{\mathbb{F}}(G)$, G a graph, $\mathbb{F}$ a field, defined as the smallest possible dimension of a projective representation of G. We showed that if $|\mathbb{F}| \geq m$, m the number of edges, then the projective dimension is $\leq 2\deg_{\max}$, twice the maximum degree of the graph. We did it by associating to every edge of a graph a vector in a vector space of dimension $2\deg_{\max}$ in such a way, that the vectors are in general position. Then we assign to each vertex the subspace spanned by the vectors corresponding to edges attached to that vertex. If two vertices are adjacent, then the spaces corresponding to them both contain the vector of the edge joining them. If we take the vectors to be in general position in a space of dimension $2\deg_{\max}$, then any $2\deg_{\max}$ of them will be linearly independent. So we get a representation. Recall that the vectors of the form $\{(1, \alpha, \alpha^2, \dots, \alpha^{k-1}) : \alpha \in \mathbb{F}\}$ are in general position in $\mathbb{F}^k$.

Theorem 18.2. *For any field $\mathbb{F}$, almost all graphs have projective dimension at least $c\sqrt{\frac{n}{\log n}}$.*

"Almost all" means the lower bound is true for all but $o(2^{\binom{n}{2}})$ graphs on a given set of n vertices, so the probability that a random graph on n vertices does not have this property tends to 0 as $n \to \infty$.

It is an open question whether almost all graphs have large projective dimension over every field. Large here means greater than n^c, where $c > 0$ is a constant. Not even $(\log n)^{1+\varepsilon}$ is known.

For the proof, we will need the theorem on zero-patterns. Recall that given m polynomials $\underline{f} = (f_1, \dots, f_m)$ of degree $\leq d$ in n variables over the field $\mathbb{F}$, then by substituing $\alpha \in \mathbb{F}^n$

in them, we get an element in $\mathbb{F}^m$. Replacing nonzero entries with a star gives us a "zero pattern," i.e., a string in $\{0,*\}^m$. Let $Z(\underline{f})$ be the number of zero-patterns for $\underline{f}$.

Theorem 18.3 (RBG(2000)). *If $d \geq 2$, then $Z(\underline{f}) \leq \binom{md}{n} < \left(\frac{emd}{n}\right)^n$.*

The bound $\left(\frac{emd}{n}\right)^n$ holds for $d = 1$ also. We will use this to estimate the projective dimension of a random graph.

Lemma 18.4. *If $\operatorname{pdim}_{\mathbb{F}}(G) = k$ and $|\mathbb{F}|$ is large enough, then there exists a projective representation over $\mathbb{F}$ of dimension $2k$ such that for every i, the dimension of every subspace representing a vertex has dimension k.*

Proof: Let U_i correspond to the vertex i. Each of them has dimension at most k. Add new vectors that are linearly independent (do it in a large space) such that each space has dimension k. Then the dimension of this space is $\leq k + nk$. Now each U_i has dimension k, and the intersections are still the same.

Exercise 18.5. The new U_i represent G.

We need to make a random projection to a $2k$ dimensional space. Suppose W is a space over $\mathbb{F}$, $\mathbb{F}$ not too small, and let $U_1, \ldots, U_n \leq W$, $\dim U_i = k$ and let S be a space $\dim S = 2k$. Let $\phi : W \to S$ be a random map (i.e. a random $2k \times \dim W$ matrix). Then the dimension of $f(U_i)$ remains the same, and the $\dim \phi(U_i) \cap \phi(U_j) = \dim \phi(U_i \cap U_j)$. i.e. these spaces avoid the kernel of the map, whose dimension is $\dim W - 2k$. Any random k-dimensional subspace, is likely to avoid it. It is also likely to avoid every $U_i + U_j$, which guarantees that the dimensions of the pairwise intersections of the U_i and the $\phi(U_i)$ are the same.

Lemma 18.6. *If $\mathbb{F}$ is large enough, then given W, U_i as above, there exists $K \leq W$ of dimension $\dim W - 2k$ such that $K \cap (U_i + U_j) = \{0\}$*

Exercise 18.7. Find out how large $|\mathbb{F}|$ has to be.

Project using this "generic" subspace as a kernel, onto a $2k$-dimensional space. This gives the desired representation, proving Lemma **??**.

Suppose now that we look for a projective representation as in the Lemma. Every subspace of it can be represented by a $k \times 2k$ matrix, where the rows form a basis of this subspace. Then the intersection condition is that if we put the two matrices on top of one another, then the resulting matrix is singular if and only if the corresponding vertices are adjacent. Now if we are looking for such a projective representation, then all entries are variables $t_{i,r,s}$. The determinants of the above matrices, (that we want to be 0 or not zero according to adjacency of the corresponding vertices), are polynomials in the $t_{i,r,s}$, where $1 \leq i \leq n$, $1 \leq r \leq k$, $1 \leq s \leq 2k$. So we have $2k^2 n$ variables. The conditions are that certain polynomials are zero in some cases and nonzero in some other cases. So this corresponds to a zero-pattern of these

$\binom{n}{2}$ polynomials. A zero entry corresponds to an edge, and $*$ corresponds to a non-edge. So the number of distinct graphs that can be represented in such a space is exactly equal to the number of zero-patterns.

Suppose at least an ε fraction of the $2^{\binom{n}{2}}$ graphs can be so represented. Then

$$\left(\frac{\mathrm{e}\binom{n}{2}2k}{2k^2n}\right)^{2k^2n} \geq Z\left(\underline{f} \geq \varepsilon 2^{\binom{n}{2}}\right), \text{so}$$

$$\left(\frac{\mathrm{e}n^2k}{2k^2n}\right)^{4k^2n} > \varepsilon 2^{n(n-1)},$$

$$\left(2\frac{n}{k}\right)^{4k^2n} > \varepsilon^2 2^{n(n-1))},$$

$$(2n)^{4k^2} > \varepsilon^{2/n} 2^{n-1}.$$

Taking logs, we get that $4k^2 \log_2(2n) > (n-1) - \frac{2}{n}\log(1/\varepsilon)$ so that

$$4k^2 > \frac{n-1}{1+\log_2 n} - \frac{2}{n}\log(1/\varepsilon).$$

Assuming $\varepsilon > c^{-n}$, we get that asymptotically $k \gtrsim \frac{1}{2}\sqrt{\frac{n}{\log n}}$.

This finishes the proof of the Theorem 18.2.

Over the real numbers, there is a much stronger result. There we have the concept of a sign-pattern. Again given polynomials as above, whenever we plug in something and get all nonzero numbers, then we get a sign pattern by looking at whether the number is positive or negative. Then we have the theorem:

Theorem 18.8 (Warren(1968)). *The number of sign patterns is* $< \left(\frac{4\mathrm{e}md}{n}\right)^n$.

(As before, m is the number of the polynomials, d is the degree, and n is the number of variables.) An application of this: If you cut up $\mathbb{R}^n$ with hypersurfaces, each region of the complement corresponds to a sign-pattern.

2 Theorem of Milnor-Thom

A real algebraic set is the set of common roots of a set of polynomials in real affine space.

Theorem 18.9 (Milnor (1964), Thom(1965)). *Let* $V \subseteq \mathbb{R}^n$ *be the set of common roots of the polynomials* $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$, *where* $\deg f_i \leq d$. *Then the number of connected components of* V *is* $\leq d(2d-1)^{n-1} \leq (2d)^n$.

In fact, the same bound holds for the sum of the Betti numbers, which gives much more (i .e, it counts higher-dimensional holes). There are computer science applications of these higher Betti numbers too. Notice that the bound does not depend on the number of polynomials, but it depends exponentially on the number of variables.

Exercise 18.10. Show Warren's theorem using the Milnor-Thom Theorem.

Exercise 18.11. From Warren's theorem deduce the theorem on number of zero-patterns for the reals and complex numbers, or any field of characteristic 0 (with a different constant than e).