1 0, 1-measures

**Exercise 22.1.** Prove the following theorem. (*Hint:* Use Zorn’s Lemma.)

**Theorem 22.2.** If $\Omega$ is an infinite set, then there exists a finite additive nontrivial 0, 1 measure on $2^{\Omega}$, $\mu : 2^{\Omega} \to \{0, 1\}$, such that $\mu(\{a\}) = 1$. Nontrivial means that $\mu(\{a\}) = 0$ for all $a$.

This is a hint to the infinite switch problem.

2 Sign-Rigidity (Paturi-Simon)

Let $M$ be a sign matrix, i.e., a matrix consisting of $\pm 1$ entries $A$ matrix $A$ of the same dimensions as $M$ realizes $M$ if the sign of $a_{i,j}$ is equal to $m_{i,j}$. The sign-rank of $M$ is the minimum rank of a matrix realizing $M$.

**Exercise 22.3 (Alon-Frankl-Rödl, 1984).** Use Warren’s theorem to prove that almost all $n \times n$ sign-matrices have sign-rank greater than or equal to $\frac{n}{2}$. 

Finding explicit matrices that satify the preceding example is a hard problem. No particular examples are known.

**Conjecture 22.4.** Hadamard matrices have sign-rank $\geq cn$.

For applications, all we need would be an explicit matrix with rank greater than $n^\epsilon$ for some fixed $\epsilon > 0$. The best known explicit bound was $\Omega(\log n)$ until 2002.

**Theorem 22.5 (Forster 2002).** Let $X \subseteq \mathbb{R}^k$ such that $|X| \geq k$ and the elements of $X$ are in general position (i.e., $k$-wise linearly independent). Then there exists an invertible $k \times k$ matrix $A$ such that $\sum_{x \in X} \frac{(Ax)(Ax)^T}{(Ax)^T(Ax)} = aI_k$, a constant multiple of the identity matrix.
If we assume that the theorem is true, the value of $a$ can easily be found by taking the trace on both sides (recall that the trace of the product of two matrices is independent of the order). This gives $a_k = \sum_{x \in X} 1 = |X|$

**Exercise 22.6 (Forster).** If $A$ realizes $M$, then $\text{rk}(A) \geq \frac{n}{\|M\|}$. Recall that $\|M\| = \max_{\|x\|=1} \|Mx\| = \sqrt{\lambda_{\max}(M^T M)}$ by the spectral theorem.

**Corollary 22.7.** If $H$ is a Hadamard matrix, then $\text{sign-rank}(H) \geq \sqrt{n}$.

**Proof:** If $A$ realizes $H$, then $\text{rk}(A) \geq \frac{n}{\|H\|}$. Since $H^T H = nI$, we get that $\|H\|^2 = \lambda_{\max}(nI) = n$.