On intersecting hypergraphs

Barry Guiduli [a,1] and Zoltán Király [b,2]

[a] *Department of Mathematics, University of Chicago, 5734 University Ave, Chicago, IL 60637. E-mail:* `bdg@cs.uchicago.edu`

[b] *Department of Computer Science, Eötvös University, Múzeum körút 6–8., Budapest, H-1088. E-mail:* `kiraly@cs.elte.hu`

Abstract

We investigate the following question: "Given an intersecting multi-hypergraph on n points, what fraction of edges must be covered by any of the best 2 points?" (Here "best" means that together they cover the most.) We call this $M_2(n)$. This is a special case of a question asked by Erdős and Gyárfás [1] (they considered r–wise intersecting and the best t points), and is a generalization of work by Mills [6], who considered the best single point.

These are very hard to calculate in general; we show that determining $M_2(q^2+q+1)$ proves the existence or nonexistence of a projective plane of order q. If such a projective plane exists, we conjecture that $M_2(q^2+q+2) = M_2(q^2+q+1)$. We further show that $M_2(q^2+q+3) < M_2(q^2+q+1)$ and conjecture that $M_2(n+2) < M_2(n)$ for all n.

We determine the specific values for $n \leq 10$. In particular we have the surprising result that $M_2(7) = M_2(8)$, leading to the conjecture made above. We further conjecture that $M_2(11) = 5/8$ and $M_2(12) = 7/12$.

To better study this problem, we introduce the concept of fractional matchings and coverings of order 2.

Key words: Hypergraphs, matching, covering.

---

[1] Research conducted while the author was visiting the Mathematical Research Institute of the Hungarian Academy of Sciences, Budapest.

[2] Research supported by OTKA grants F 014919, T 14302, T 19367 and T 17580.

1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair (V, E) where $V = V(\mathcal{H})$ is a (finite) set called the vertex (or point) set, and $E = E(\mathcal{H})$ is a (finite) collection of subsets of V called the edge-set. If $\mathcal{H}$ contains multiple edges, then it is called a *multihypergraph*. The *dual* of $\mathcal{H}$ is the hypergraph $\mathcal{H}^T$ obtained by interchanging the roles of vertices and edges. I.e. $V(\mathcal{H}^T) = E(\mathcal{H})$ and $E(\mathcal{H}^T) = \{E(v) : v \in V(\mathcal{H})\}$ where $E(v) = \{X \in E(\mathcal{H}) : v \in X\}$. Clearly $(\mathcal{H}^T)^T \cong \mathcal{H}$. $\mathcal{H}$ is called *intersecting* if every pair of edges intersects nontrivially; it is called *r-wise intersecting* if every r-tuple of edges intersects nontrivially. The set of edges covered by a vertex v is the set $\{X \in E(\mathcal{H}) : v \in X\}$, and the set of edges covered by a set of vertices is the union of the sets covered by each vertex.

The following problem was asked by Erdős and Gyárfás [1]:

Problem 1 *Let $\mathcal{H}$ be an r-wise intersecting multihypergraph on n vertices. What fraction of edges must be covered by the "best" t points?*

By the word "best" above, we understand the t points that cover the largest fraction of edges. Erdős and Gyárfás remarked that for $n \leq rt$, the best ratio is 1, i.e. suitable t points cover everything. They also proved that for $n = rt+1$, the best ratio is $1 - 1/\binom{rt+1}{r}$. In particular, this shows for $n = 5$, $t = 2$, $r = 2$, that the ratio is $9/10$. The first unknown case was $n = 6$, $t = 2$, $r = 2$.

The problem arises as a generalization of a problem studied by Mills [6]. We state this here in the dual, as it was studied by Mills. We say that a family $\mathcal{F}$ of subsets of the set M covers the pairs of M if $\forall a, b \in M$, there is an $F \in \mathcal{F}$ such that $a, b \in F$. This is equivalent to its dual being intersecting.

Problem 2 *If $\mathcal{F}$ is a family of n subsets of an m-element set which covers the pairs, how large must the largest set be compared to m?*

In the above notation of Erdős and Gyárfás this is the case $r = 2$, $t = 1$. This quantity, which we shall denote $M_1(n)$ and call the Mills Number was determined by Mills [6] for $n \leq 13$. Füredi [2] determined some nice general result for M_1 (see [3] and [4]). Pach and Surányi [7] showed that $M_1(q^2+q+1) = (q+1)/(q^2+q+1)$ if and only if there exists a projective plane of order q.

In this paper we investigate the case $r = 2$, $t = 2$. To make the notion of "fraction of edges" more precise and easier to handle, we introduce a weight function. Let $wt\colon E \to \mathbb{R}$ be a nonnegative normalized weight function; this means that $wt(X) \geq 0$ and $\sum_{X \in E} wt(X) = 1$. (For example the weight of a subset of the vertices could be the number of times this set occurs as an edge in the multihypergraph, divided by the total number of edges.) In this way, we need only consider hypergraphs (those with no multiple edges). Although the weights may be irrational, it is clear that for the optimal case, the weights will all be rational, and in this way we see that these two formulations are equivalent.

Furthermore, we define the weight covered by a single vertex v to be

$$wt(v) := \sum_{v \in X \in E} wt(X),$$

and the weight covered by two vertices u and v to be

$$wt(u \vee v) := \sum_{\substack{X \cap \{u,v\} \neq \emptyset \\ X \in E}} wt(X)$$

With these definitions, we can better define the quantity studied. For every positive integer n, the *Mills Number* (also called the First Mills Number) is defined by

$$M_1(n) := \min_{\mathcal{H}} \min_{wt} M_1(\mathcal{H}, wt),$$

where the first minimum is taken over all intersecting hypergraphs on n vertices, the second one is over all nonnegative normalized weight functions on $E(\mathcal{H})$ and

$$M_1(\mathcal{H}, wt) := \max_{v \in V} wt(v).$$

Similarly, the Second Mills Number is

$$M_2(n) := \min_{\mathcal{H}} \min_{wt} M_2(\mathcal{H}, wt),$$

$$M_2(\mathcal{H}, wt) := \max_{u,v \in V} wt(u \vee v).$$

We have the following results for M_2:

Theorem 3 *Let $n = q^2 + q + 1$. If there exists a projective plane of order q, then $M_2(n) = (2q+1)/n$, otherwise $M_2(n) > (2q+1)/n$.*

Theorem 4 *The following are the exact values of the Second Mills Number*

n:	4	5	6	7	8	9	10
$M_2(n)$:	1	9/10	4/5	5/7	5/7	9/13	2/3

We mention here that for $n = 4$ the calculation is trivial and for $n = 5$ this follows from the more general result of Erdős and Gyárfás [1] mentioned above.

2 Coverings and Matchings

A *matching* is a subfamily of pairwise disjoint edges. The matching number $\nu(\mathcal{H})$ is the maximum number of edges in a matching of $\mathcal{H}$. A *cover* of $\mathcal{H}$ is a subset $T \subset V$ which meets all the edges of $\mathcal{H}$, and the covering number $\tau(\mathcal{H})$ is the size of the smallest cover of $\mathcal{H}$.

A *fractional matching* in a hypergraph $\mathcal{H}$ is a nonnegative function on the edges $w : E \to \mathbb{R}$, such that

$$\sum_{X \ni v} w(X) \leq 1 \qquad \forall v \in V.$$

The value of w, written $|w|$, is the total sum $\sum_{X\in E} w(X)$. The *fractional matching number* of $\mathcal{H}$, written $\nu^*(\mathcal{H})$ is defined as the largest value of a fractional matching. A *fractional cover* of $\mathcal{H}$ is a nonnegative function on the vertices $t : V \to \mathbb{R}$, such that

$$\sum_{v \in X} t(v) \geq 1 \qquad \forall X \in E.$$

The value of t, written $|t|$, is the total sum $\sum_{v\in V} t(v)$. The *fractional covering number* of $\mathcal{H}$, written $\tau^*(\mathcal{H})$ is defined as the smallest value of a fractional cover. By the Duality Theorem of linear programming, we have

$$\nu(\mathcal{H}) \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq \tau(\mathcal{H}).$$

It can easily be seen by scaling of the weight function, that the First Mills Number satisfies

$$M_1(n) = \frac{1}{\nu^*(n)}$$

where $\nu^*(n) = \max \nu^*(\mathcal{H})$, the maximum taken over all intersecting hypergraphs on n vertices.

For the purposes of investigating the Second Mills Number, we introduce fractional matchings and coverings of order 2. A *fractional matching of order* 2 in a hypergraph $\mathcal{H}$ is a nonnegative function on the edges $w_2 : E \to \mathbb{R}$, such that

$$\sum_{X \cap \{u,v\} \neq \emptyset} w_2(X) \leq 1 \qquad \forall u, v \in V.$$

The value of w_2, written $|w_2|$, is the total sum $\sum_{X\in E} w_2(X)$. The *fractional matching number of order* 2 of $\mathcal{H}$, written $\nu_2^*(\mathcal{H})$ is defined as the largest value of a fractional matching of order 2. A *fractional cover of order* 2 of $\mathcal{H}$ is a function on the pairs of vertices $t_2 : \binom{V}{2} \to \mathbb{R}$ such that

$$\sum_{\{u,v\}\cap X \neq \emptyset} t_2(\{u,v\}) \geq 1, \qquad \forall X \in E.$$

The value of t_2, written $|t_2|$, is the total sum $\sum_{\{u,v\}\in\binom{V}{2}} t_2(\{u,v\})$. The *fractional covering number of order* 2 of $\mathcal{H}$, written $\tau_2^*(\mathcal{H})$ is defined as the smallest value of a fractional cover of order 2. Computing these two quantities are also dual linear programming problems, and by the Duality Theorem we conclude that $v_2^*(\mathcal{H}) = \tau_2^*(\mathcal{H})$. Again, by scaling the weights, we see that

$$M_2(n) = \frac{1}{\nu_2^*(n)}$$

where $\nu_2^*(n) = \max \nu_2^*(\mathcal{H})$, the maximum taken over all intersecting hypergraphs on n vertices. For every hypergraph $\mathcal{H}$, it is easy to see that $\nu_2^*(\mathcal{H}) \leq \nu^*(\mathcal{H})$ and $\tau^*(\mathcal{H}) \leq 2\tau_2^*(\mathcal{H})$ (if t_2 is a fractional cover of order 2 then $t(v) := \sum_u t_2(\{u,v\})$ is a fractional cover with value $2|t_2|$) it follows that $M_1(n) \leq M_2(n) \leq 2M_1(n)$ for all n.

Conjecture 5 *$M_2(n) < 2M_1(n)$ for all n.*

3 General results

We start this section by giving a general lower bound for the Second Mills Number. This lower bound is tight in the cases where $n = 4$, 5, 6, 7, and 10. In addition, it is tight whenever $n = q^2 + q + 1$ and there is a projective plane of order q. Let

$$L(k,n) := \min\left(\frac{2}{k-1}, \frac{k(2n-k-1)}{n(n-1)}\right).$$

and

$$L(n) := \max_{\substack{k\in\mathbb{N}\\ 1<k\leq n}} L(k,n)$$

Theorem 6 *For every $n > 2$ the following inequality holds:*

$$M_2(n) \geq L(n).$$

Proof: The proof uses repeated applications of the Pigeon Hole Principle. Let $\mathcal{H}$ be an (arbitrary) intersecting hypergraph on n vertices, wt a nonnegative normalized weight function on $E(\mathcal{H})$ and $1 < k \leq n$ an integer. It is enough to show that $M_2(\mathcal{H}, wt) \geq L(k,n)$. In particular, if there is an edge of size $\leq k-1$

in $\mathcal{H}$, then the best two points from this edge cover at least $2/(k-1)$. Actually as we may assume that the weight of this edge is nonzero, the best two points cover strictly more then $2/(k-1)$. Otherwise every edge has size at least k and we show that there are two vertices that cover at least $k(2n-k-1)/n(n-1)$. By the normalization of the weight function, we have

$$\sum_x wt(x) \geq k.$$

Hence there is a v such that $wt(v) \geq k/n$. We now consider the hypergraph obtained by removing vertex v and all edges containing it. The best vertex u here covers at least

$$\frac{k(1-wt(v))}{n-1}.$$

Together, u and v cover at least

$$\begin{aligned} wt(u \vee v) \geq wt(v) + \frac{k(1-wt(v))}{n-1} = \frac{wt(v)(n-1-k)+k}{n-1} \geq \\ \geq \frac{kn-k-k^2+nk}{n(n-1)} = \frac{k(2n-k-1)}{n(n-1)}. \end{aligned}$$

This completes the proof. $\square$

We give an alternate proof in order to make the language of fractional coverings more familiar to the reader. Let $\mathcal{H}$ be an (arbitrary) intersecting hypergraph on n vertices and $k > 1$ be an integer. If there is an edge X of size $k-1$, then the function t_2 defined to be $t_2(u,v) = \frac{1}{k-2}$ for $u, v \in X$ and 0 otherwise is a good fractional cover of order 2 with value $|t_2| = \frac{k-1}{2}$, so $\tau_2^*(\mathcal{H}) \leq \frac{k-1}{2}$. If no edge has size $k-1$ but there is an edge with smaller size take X to be a set of size $k-1$ containing the smaller edge and the same argument works. Otherwise all edges must have size $\geq k$. In this case define t_2 to be $t_2(u,v) = \frac{2}{k(2n-k-1)}$ for all $u, v \in V(\mathcal{H})$. This is a good fractional cover of order 2 with value $|t_2| = \frac{n(n-1)}{k(2n-k-1)}$, so $\tau_2^*(\mathcal{H}) \leq \frac{n(n-1)}{k(2n-k-1)}$ and this completes the second proof.

We will use the function $L(n)$ in the next section, with specific hypergraphs to show the exact value of M_2 for some small values of n. Here instead we give a few more general results.

Pach and Surányi [7] showed that

Theorem *Let $n = q^2+q+1$. If there exists a projective plane of order q, then $M_1(n) = (q+1)/n$, otherwise $M_1(n) > (q+1)/n$.*

We use this result to prove the analogous statement for M_2, which we mentioned in the Introduction, and repeat here:

Theorem 1 *Let $n = q^2 + q + 1$. If there exists a projective plane of order q, then $M_2(n) = (2q + 1)/n$, otherwise $M_2(n) > (2q + 1)/n$.*

Proof: This can be proved directly without referring to the result of Pach and Surányi, but the proof is considerably shorter using their result. The lower bound proved above gives $M_2(n) \geq L(q+1, n) = (2q+1)/n$. If there is a projective plane on n points, then $M_2(n) = (2q + 1)/n$ because the projective plane achieves this bound. Assume that $M_2(n) = (2q + 1)/n$ and let $\mathcal{H}$ be a hypergraph and wt a weight function attaining this bound. Because equality holds here, equality must hold in each of the inequalities in the proof of the lower bound corresponding to the case $k = q+1$. It follows that the first point covers $(q + 1)/n$, thus $M_1(\mathcal{H}, wt) = (q + 1)/n$. Now we may use the result of Pach and Surányi to conclude that $M_1(n) = (q + 1)/n$ and so there must be a projective plane of order q. $\square$

From the lower bound and the density of prime powers, we get the asymptotic result that

Theorem 7 $M_2(n) \approx 2/\sqrt{n}$.

One of the interesting results to follow in the next section is that $M_2(8) = M_2(7) = 5/7$. We conjecture that this is true whenever there is a projective plane, namely:

Conjecture 8 *If $n = q^2 + q + 1$ and there exists a projective plane of order q, then $M_2(n + 1) = M_2(n)$.*

We show here that this is not the case for $n + 2$.

Theorem 9 *If $n = q^2 + q + 1$, $q > 1$, and there exists a projective plane of order q, then $M_2(n + 2) \leq (4q^2 - 2q - 3)/(2q^3 - 3) < M_2(n)$.*

Proof: We construct a hypergraph on $n + 2 = q^2 + q + 3$ points adding two points to the projective plane on n points. Let x be a point of the projective plane. We essentially blow x up into a projective plane on three points. Specifically, we replace x by x_1, x_2, x_3, and every edge X that passed through x is replaced by three new edges, X_1, X_2, X_3, where X_i contains all the points from X excluding only x, and of the three new points only misses x_i. All of the edges not passing through x are unchanged. This new hypergraph is intersecting, and there are two types of edges, Type I which go through two of the x_i's and Type II which do not. These will get different weights. Let

$$wt(Y) = \begin{cases} \frac{t-1}{2q^3-3} & \text{if } Y \text{ is Type I,} \\ \frac{2t-3}{2q^3-3} & \text{if } Y \text{ is Type II.} \end{cases}$$

It is easy to check that the best two points cover $(4q^2 - 2q - 3)/(2q^3 - 3)$, which for $q > 1$ is less than that in the projective plane (here any two points cover: $(2q+1)(q^2+q+1)$). $\square$

We conjecture that this is the case in general:

Conjecture 10 *$M_2(n+2) < M_2(n)$ for all n.*

4 Results for small n

We mentioned earlier that for the cases $n = 4$, 5, 6, 7, and 10, the lower bound $L(n)$ is achieved. $n = 8$ and $n = 9$ will follow in the next section as they require a more sophisticated lower bound. $n = 7$ corresponds to the Fano Plane PG(2,2), and is briefly discussed with all projective planes and is omitted here. For the other cases, it is enough to give an example of a hypergraph and a weight function so that the weight covered by any two points is equal to the lower bound. In the examples to follow, all edges have equal weight. The matrices given below are incidence matrices of the hypergraphs, there are 0-1 matrices, with rows are indexed by vertices and columns by edges. A 1 in position i, j indicates that vertex i lies on edge j. We use a dash to indicate a zero entry.

For $n = 4$, $L(4) = 1$, so there is nothing to prove, any two points of an intersecting hypergraph on 4 vertices cover all the edges.

$M_2(5) = L(5) = \frac{9}{10}$. The example is the hypergraph on 5 points with all 3-sets for edges, equally weighted. This is clearly intersecting and every pair of points misses one edge, so each pair covers 9/10.

$M_2(6) = L(6) = \frac{4}{5}$. The example is unique and is the two-graph of the icosahedron [8]. More precisely, consider a 5-cycle plus an isolated point. From this construct a 3-regular hypergraph where the edges are all 3-sets of vertices that contain one edge in the pentagon. There are then two types of edges, either a pair of adjacent vertices and the opposite vertex of the 5-cycle, or a pair of adjacent vertices and the isolated point. The hypergraph is clearly intersecting, and any two points miss 2 of the 10 edges.

$$\begin{bmatrix} 1&1&1&1&1&-&-&-&-&- \\ 1&1&-&-&-&1&1&1&-&- \\ -&-&1&1&-&1&1&-&1&- \\ 1&-&1&-&-&-&-&1&1&1 \\ -&1&-&-&1&-&1&-&1&1 \\ -&-&-&1&1&1&-&1&-&1 \end{bmatrix}$$

$M_2(10) = L(10) = \frac{2}{3}$. The hypergraph achieving this is not unique. We present one here that can be described very nicely. Consider a 3×3 grid of points $x_{i,j}$ $(1 \leq i, j \leq 3)$. Add one more additional point y. Consider the 6 sets that go through y and either a row or column of the grid, and the 9 sets that arise from taking the union of a row and a column from the grid, and excluding the intersection point. This is intersecting and any pair of points covers $2/3$ of the edges. We give the incidence matrix:

$$\begin{bmatrix} 1&1&1&1&1&1&-&-&-&-&-&-&-&-&- \\ 1&-&-&1&-&-&-&1&1&1&-&-&1&-&- \\ 1&-&-&-&1&-&1&-&1&-&1&-&-&1&- \\ 1&-&-&-&-&1&1&1&-&-&-&1&-&-&1 \\ -&1&-&1&-&-&1&-&-&-&1&1&1&-&- \\ -&1&-&-&1&-&-&1&-&1&-&1&-&1&- \\ -&1&-&-&-&1&-&-&1&1&1&-&-&-&1 \\ -&-&1&1&-&-&1&-&-&1&-&-&-&1&1 \\ -&-&1&-&1&-&-&1&-&-&1&-&1&-&1 \\ -&-&1&-&-&1&-&-&1&-&-&1&1&1&- \end{bmatrix}$$

5 Nontrivial lower bounds

5.1 General approach

In order to get sharp results we sometimes need nontrivial lower bounds for M_2. The general approach for this is the following.

Let $\mathcal{S}$ be the family of all intersecting hypergraphs on n points. Suppose it is divided into some (not necessarily disjoint) subfamilies:

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

Further suppose that we have functions $t_i : \binom{\{1\ldots n\}}{2} \to \mathbb{R}$ $(i = 1\ldots k)$ such that t_i is a good fractional cover of order 2 for all hypergraphs in the family $\mathcal{S}_i$ (in this section every cover will be of order 2). Then clearly

$$M_2(n) \geq \frac{1}{\max(|t_i|)}.$$

In order to use this idea we have to find subdivisions and appropriate fractional covers. Two examples are shown for this technique in the next subsections. The language we used is slightly different from that above, but the idea is the same.

5.2 The case $\mathbf{n} = \mathbf{8}$

Theorem 11 $M_2(8) = 5/7$

Proof: That $M_2(8) \leq 5/7$ is obvious since the Fano plane ($PG(2,2)$) is a good construction for it. We prove that $M_2(8) \geq 5/7$ in an indirect way. Suppose there is an intersecting edge-weighted hypergraph $\mathcal{H}$ on 8 vertices, say $V = \{1,2,3,4,5,6,7,8\}$, such that $wt(x \vee y) < 5/7$ for all $x, y \in V$, which means that $\tau_2^*(\mathcal{H}) > 7/5$. We will get contradictions for families of possible hypergraphs by showing a fractional cover of order two for that family which has value $\leq 7/5$. We will define a set of hypergraphs, exhibit a fractional cover of order two, and then show that it has weight $\leq 7/5$, showing that $\mathcal{H}$ is not in this set. We then proceed, considering all those intersecting hypergraphs on 8 vertices for which we have not yet given a covering or order 2, eventually ruling out all hypergraphs. When giving the values of such a cover t we will write $t(x,y)$ instead of $t(\{x,y\})$ and will give only the non-zero values. As the hypergraphs are unlabeled, we can label each to our advantage, for instance, when considering a hypergraph with an edge of size 3, we may assume that this edge is $\{1,2,3\}$.

Case i. $\mathcal{H}$ contains a two-set ($\{1,2\}$) or a one-set ($\{1\}$). Let $t(1,2) = 1$. As $\mathcal{H}$ is intersecting, t is clearly a good fractional cover and $|t| = 1 < 7/5$.

We conclude from this that $\mathcal{H}$ does not have any two-sets or one-sets.

Case ii. All edges of $\mathcal{H}$ have at least 4 elements. Let $t(x,y) = 1/22$ for all $1 \leq x < y \leq 8$. This is a good fractional cover for $\mathcal{H}$ and $|t| = 28/22 < 7/5$.

Hence $\mathcal{H}$ must have a three-set $A = \{1, 2, 3\}$.

Case iii. There are no three-sets in $\mathcal{H}$ meeting A in one point. Let $t(x, y) = 1/3$ whenever $x, y \in A$ and $t(x, y) = 1/27$ whenever $x, y \notin A$. We state that t is a good fractional cover for $\mathcal{H}$. The three-sets of $\mathcal{H}$ meet A in at least two points, so they get at least 1 from t. The other sets of $\mathcal{H}$ contain at least four points and either meet A in more than one point, in this case they get at least 1, or meet A in one point and get at least $2/3 + 9/27 = 1$. As $|t| = 3/3 + 10/27 < 7/5$ we can conclude that there must be a three-set $B = \{3, 4, 5\}$ meeting A in one point.

Case iv. There are no three-sets in $\mathcal{H}$ meeting both A and B in one point different from 3. Let $t(x, y) = 1/5$ whenever $x = 3$ and $y \in \{1, 2, 4, 5\}$ or $\{x, y\} = \{1, 2\}$ or $\{4, 5\}$. Let $t(x, y) = 1/15$ whenever $x, y \notin A \cup B$. This is a good fractional cover for $\mathcal{H}$ as the three-sets of $\mathcal{H}$ either meet $A \cup B$ in $\{3\}$, in which case they get $4/5 + 3/15 = 1$, or they meet $A \cup B$ in at least two points, in which case they cover at least $5/5$ (we are assuming that this set does not intersect A and B in one point each, different from $\{3\}$). The other sets of $\mathcal{H}$ have at least four elements and get at least $4/5 + 3/15 = 1$ or $5/5 = 1$, using the intersecting property. $|t| = 6/5 + 3/15 = 7/5$.

We now know that there is another three-set $C = \{1, 5, 6\} \in \mathcal{H}$.

Case v. There are three three-sets in $\mathcal{H}$: A_1, A_2, A_3 that all intersect pairwise in the same point, say a. We call this a 3-3-star. Let $t(x, y) = 2/15$ whenever $x = a$ and $x \neq y \in \cup_i A_i$ and $t(x, y) = 1/5$ whenever $a \neq x, y \in A_i$ for some i. This is a good fractional cover for $\mathcal{H}$. As there are eight points and all sets have size at least three, every set of $\mathcal{H}$ intersects $\cup_i A_i$ in at least two points. If a set contains a then it gets at least $12/15 + 1/5 = 1$, otherwise it contains at least one point different from a from all three sets and gets at least $6/15 + 3/5 = 1$. $|t| = 12/15 + 3/5 = 7/5$.

We now conclude that $\mathcal{H}$ does not contain a 3-3-star.

Case vi. At this point we know that $\mathcal{H}$ does not contain one- and two-sets, it contains A, B, C defined above and there are no 3-3-stars in it. For simplicity call the points $1, 3, 5$ vertex-points, $2, 4, 6$ edge-points and $7, 8$ outer points. We call a vertex-point and an edge-point *opposite* if no set from A, B, C contains both. Let $t(x, y) = 1/10$ whenever x and y are vertex-points, $t(x, y) = 3/20$ whenever x is a vertex-point and y is a non-opposite edge-point, and $t(x, y) = 1/60$ if exactly one of x, y is an outer point. We want to show that t is a good fractional cover for $\mathcal{H}$. Let D be a set of $\mathcal{H}$ and $E = D \cap (A \cup B \cup C)$. If E contains the three edge-points or the three vertex-points then E itself gets at least $18/20 + 6/60 = 1$. If E contains two vertex-points then either E contains at least one edge-point as well and gets at least $3/10 + 12/20 + 6/60 = 1$ or D contains at least one outer point and gets at least $3/10 + 12/20 + 8/60 > 1$. If E

contains one vertex-point then either E contains two edge-points (one of them must be opposite to the vertex-point) and gets at least $2/10+15/20+6/60 > 1$ or D contains the opposite edge-point and the two outer points (there is no 3-3-star) and gets at least $2/10 + 12/20 + 12/60 = 1$. So t is a good vertex cover of $\mathcal{H}$ and $|t| = 3/10 + 18/20 + 12/60 = 7/5$.

With this last case, we have competed the proof. $\square$

5.3 The case $\mathbf{n = 9}$

Theorem 12 $M_2(9) = 9/13$

Proof: That $M_2(9) \leq 9/13$ follows from Theorem 9. We should mention that in the construction given in the proof of that theorem, the weights will be equal for this case ($q = 2$) and the hypergraph achieving 9/13 can be described as follows:

$$\begin{bmatrix} 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & - & - & - \\ 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - \\ 1 & - & - & 1 & - & - & - & - & - & - & 1 & 1 & 1 \\ - & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & 1 \\ - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - \\ - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\ - & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - \\ - & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\ - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \end{bmatrix}$$

The proof of the lower bound is very similar to the proof given above. We have the same cases, with only one new one added between Case iv and Case v. We give only the differences here. We want to show fractional covers of order two with value $\leq 13/9$.

Case ii. Use weights 1/26 instead of 1/22.

Case iii. Use weights 1/36 instead of 1/27.

Case iv. Use weights 1/25 instead of 1/15.

Case iv$\frac{\mathbf{i}}{\mathbf{ii}}$. There is a point a and four sets A_1, A_2, A_3, A_4 in $\mathcal{H}$ such that the intersection of any two is a. From the contradiction of the previous case we know that there must be a three-set intersecting both A_1 and A_2 in one point different from a. But this set could not intersect both A_3 and A_4, so this case does not occur.

Case v. The weights are the same but in the proof when we need that every set of $\mathcal{H}$ intersects $\cup_i A_i$ in more than one point, we prove it by using the impossibility of the previous case.

Case vi. Use weights 10/108 instead of 1/10, 15/108 instead of 3/20, and 2/108 instead of 1/60. $\square$

6 The cases $\mathbf{n = 11}$ and 12

We end with the first two open cases.

Conjecture 13 *$M_2(11) = 5/8$ and $M_2(12) = 7/12$.*

We will show that these are in fact upper bounds for the Second Mills number by presenting multihypergraphs which achieve these values.

For $n = 11$ we have

$$\begin{bmatrix}
3&3&3&3&-&-&-&-&2&2&2&-&-&-&-&-&-&-&-&- \\
-&-&-&-&3&3&3&3&2&2&2&-&-&-&-&-&-&-&-&- \\
3&-&3&-&3&-&3&-&-&-&-&2&2&2&-&-&-&-&-&- \\
-&3&-&3&-&3&-&3&-&-&-&2&2&2&-&-&-&-&-&- \\
3&-&-&3&-&3&3&-&-&-&-&-&-&-&2&2&2&-&-&- \\
-&3&3&-&3&-&-&3&-&-&-&-&-&-&2&2&2&-&-&- \\
3&-&3&-&-&3&-&3&-&-&-&-&-&-&-&-&-&2&2&2 \\
-&3&-&3&3&-&3&-&-&-&-&-&-&-&-&-&-&2&2&2 \\
-&-&-&-&-&-&-&-&2&2&-&2&2&-&2&2&-&2&2&- \\
-&-&-&-&-&-&-&-&2&-&2&2&-&2&2&-&2&2&-&2 \\
-&-&-&-&-&-&-&-&-&2&2&-&2&2&-&2&2&-&2&2
\end{bmatrix}$$

Here the numbers 2 and 3 indicate that in the multihypergraph these edges would appear twice and three times, respectively. There would be a total of 48 edges, so the weight functions would be 2/48 and 3/48, respectively.

For $n = 12$ there is a nice hypergraph called a *twisted projective plane* [3] which is the unique hypergraph having the property that all of its edges have size

[3] A twisted projective plane is a q-regular intersecting hypergraph of degree q with $q^2 + q$ vertices, $q^2 + q$ edges, and the edges cover all pairs. It is only known to exist for $q \leq 3$. Cf. [5].

4, each vertex has degree 4, and it is intersecting. It can be represented as the residues $\{0, 1, 4, 6\}$ modulo 12, which are cyclically permuted to give all edges. If can easily be verified that the best two points cover $7/12$, and that $L(12) = 19/33$.

$$\begin{bmatrix}
1 & 1 & - & - & 1 & - & 1 & - & - & - & - & - \\
- & 1 & 1 & - & - & 1 & - & 1 & - & - & - & - \\
- & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - \\
- & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - \\
- & - & - & - & 1 & 1 & - & - & 1 & - & 1 & - \\
- & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 \\
1 & - & - & - & - & - & 1 & 1 & - & - & 1 & - \\
- & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - \\
- & - & 1 & - & 1 & - & - & - & - & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & - & - & - & - & 1
\end{bmatrix}$$

For the cases $n = 11, 12$, we can try similar methods as those used above for $n = 8, 9$. We assume that there is a hypergraph that beats one of these and then we show that there are no edges of size smaller than 4; there exists an edge of size 4; for every edge X of size 4 there is another edge Y of size 4 that intersects X in one point; and for every two edges X and Y of size 4 that intersect in one point, there is an edge Z of size 4 that intersects both in one point such that X, Y, and Z have no point in common. Beyond this step, arguments similar to those for $n = 8, 9$ become very complicated.

7 Acknowledgments

We are grateful to András Gyárfás for introducing us to this problem and for many useful discussions, as well as to Zoltán Füredi for many helpful ideas. These results were presented at the *Conference on Hypergraphs and Symmetric Structures*, held in Balatonlelle, Hungary, 1995, and at the *Second Slovenian International Conference on Graph Theory and Combinatorics*, held in Bled, Slovenia, 1995, where this paper was presented. We would like to thank the organizers of these conferences. The first author would also like to thank the Mathematical Research Institute of the Hungarian Academy of Sciences for their generous hospitality during his visit.

References

[1] P. Erdős and A. Gyárfás, Personal communication, 1994.

[2] Z. Füredi, On maximal intersecting families of finite sets, *J. Combin. Theory Ser. A.* **28** (1980) 282–289.

[3] Z. Füredi, Covering pairs by q^2+q+1 sets, *J. Combin. Theory Ser. A.* **54** (1990) 282–289.

[4] Z. Füredi, Intersecting designs from linear programming and graphs of diameter two, *Discrete Math.* **127** (1993) 187–207.

[5] E.R. Lamken, R.C. Mullin, and S.A. Vanstone, Some non-existence results on twisted planes related to minimum covers *Congr. Numer.* **48** (1985) 265–275.

[6] W.H. Mills, Cover design. I. Covering by a small number of subsets, *Ars Combin.* **8** (1979) 199–315.

[7] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming, *in* "Algebraic Methods in Graph Theory (Szeged, Hungary, 1978, L. Lovász et al. Eds.)," Proc. Colloq. Math. Soc. J. Bolyai, 25, North-Holland, Amsterdam, 1981, 599-629.

[8] J.J. Seidel, A survey of two-graphs, *Proc. Intern. Coll. Teorie Combinatorie,* (Roma 1973). Acad. Naz. Lincei, Roma, 1976, 481–511.