2-partition-transitive tournaments

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Abstract

Given a tournament score sequence $s_1 \geq s_2 \geq \cdots \geq s_n$, we prove that there exists a tournament T on vertex set $\{1, 2, \ldots, n\}$ such that the degree of any vertex i is s_i and the subtournaments of T on both the even and the odd vertices are transitive in the given order. This means that i beats j whenever $i < j$ and $i \equiv j \pmod 2$.

For any score sequence, we give an algorithm to construct a tournament of the above form, i.e. it is transitive on evens and odds in the given order. This algorithm fixes half of the edges of the tournament and then is similar to the algorithm for constructing a tournament given its score sequence.

Another consequence provides asymptotics for the maximum number of edges in score unavoidable digraphs. From a result of Ryser, it is possible to get from any tournament to this special tournament by a sequence of triangle reversals. We show that $n^2/2$ reversals are always enough and that in some cases $(1 - o(1))n^2/32$ are required. We also show that such a sequence of triangle reversals can be found in $O(n^2)$ time.

1 Introduction

A tournament $T = (V, E)$ is an orientation of a complete graph, i.e. it is a directed graph such that $(x, y) \in E \iff (y, x) \notin E$. If $(x, y) \in E$, we write $x \to y$ and say that x dominates y. The *degree* of a vertex x, also called its *score* and denoted by either $d(x)$ or s_x, is the number of vertices dominated by x (the out-degree with T viewed as a directed graph). The *score sequence* of a tournament is the sequence of degrees of its vertices, given in the decreasing order. Let x be a vertex of a tournament T and S a subset of T, we denote by $d_S(x)$ the number of elements in S which are dominated by x.

The triple (a, b, c) is called a *directed triangle* (or simply a *triangle*) if $(a, b), (b, c), (c, a) \in E$. We similarly define directed cycles. A *reversal* of a directed cycle in a tournament is an operation that reverses the edges in the directed cycle. This operation does not alter the score sequence. Two tournaments T and T' defined on the same set of vertices are *cycle-reversal-equivalent* (resp. *triangle-reversal-equivalent*) if T can be transformed into T' by a succession of cycle reversals (resp. directed triangle reversals).

A tournament is called *transitive (acyclic)* if $p \to q$ and $q \to r$ imply that $p \to r$. Two vertices $a \to b$ of a transitive tournament T are *consecutive* if there is no c in T such that $a \to c$ and $c \to b$.

Definition. A tournament is 2-*partition-transitive* if there exists a partition of the set of vertices $V = A \cup B$ such that the tournaments induced on both A and B are transitive.

Let T be a tournament on set $[n] = \{1, 2, \ldots, n\}$ and further let $P = \{p_1, p_2, \ldots, p_r\} \subset [n]$ be a set such that P induces the transitive subtournament given by $p_i \to p_j$ if and only if $i < j$ (we say that P is *oriented in the indexed order*). For any point x in $T \setminus P$, we define the *dominancy word* of P on x to be $a_1a_2 \dots a_r$ where $a_i = 1$ if p_i dominates x, otherwise $a_i = 0$. Clearly, if the word of P on x does not contain 01 then the tournament on $P \cup \{x\}$ is transitive.

We recall the result of Ryser [15] concerning scores and cycle reversals:

Theorem 1 (Ryser [15]) *If two tournaments T and T' are defined on the set V and $d_T(v) = d_{T'}(v)$ for any $v \in V$ then T is triangle-reversal-equivalent to T'.*

This result implies in particular that cycle-reversal-equivalence and triangle-reversal-equivalence are the same. A tournament is *regular* if its score sequence is constant. Following Ryser's theorem all the regular tournaments on $2k+1$ vertices are cycle-reversal-equivalent. Moreover, there is only one regular tournament on $2k+1$ vertices which is 2-partition-transitive: we arrange the $2k+1$ vertices around a circle and each vertex dominates the next k vertices around the circle clockwise.

The following more general question was the starting point for our investigations. For a given tournament score sequence, does there exist a tournament having not only this score sequence but also other prescribed properties such as transitive or bipartite subtournaments of a given size? We may also ask for the maximal number of edges a digraph D_n on n vertices can have such that for each score sequence of length n there exists a tournament having the given score sequence and containing D_n as a subdigraph. Digraphs which satisfy this condition are called *score unavoidable.*

Our main result states that for a given score sequence with $s_1 \geq s_2 \geq \dots \geq s_n$ there exists a tournament T on the vertex set $\{1, \dots, n\}$ such that the degree of any vertex i is indeed s_i and the subtournaments of T on both the even and the odd vertices are transitive in the given order. This means that i dominates j whenever $i < j$ and $i \equiv j \pmod 2$. We say that T is a *balanced 2-partition transitive tournament.*

As a consequence of this result we remark that in order to construct a tournament from a given score sequence we can fix roughly half of the edges in advance. Thus we immediately obtain the lower bound $\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor$ for the maximal number of edges in a score unavoidable digraph D_n. We show that $\left\lfloor \frac{n^2}{4} \right\rfloor$ is an upper bound.

Let T be a tournament and T' be a balanced 2-partition transitive tournament with the same score sequence as T, which we showed exists. In light of Theorem A, we may ask for the minimal number of triangle reversals which are necessary to transform T into T'. If $|V(T)| = n$ then we can bound this quantity between $\frac{n^2}{2}$ above and $(1-o(1))n^2/32$ below, thus determining the order of magnitude.

2 Partition into two transitive tournaments

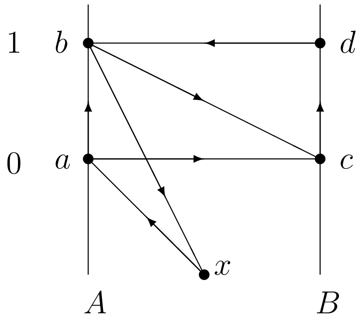
In this section we prove that for any score sequence, there is a tournament which is 2-partition transitive.

Theorem 2 *Every tournament T is cycle-reversal-equivalent to a 2-partition transitive tournament.*

Proof. We use induction on the size of T. It is clear that if T contains two vertices, then T is 2-partition-transitive. Let $|V(T)| = n$ and assume that the theorem is true for all tournaments on fewer than n points. Let x be a point in the tournament. By the inductive hypothesis, there is a sequence of cycle reversals on $T \setminus \{x\}$ which makes it 2-partition-transitive. We may assume that $T \setminus \{x\}$ is already in this form with partition $A \cup B$, and that the partition has certain extremal properties with respect to all possible tournaments in the cycle-reversal-equivalence class of T, specifically:

We consider a partition where A is minimal in size and the dominancy word of A (where the elements of A are indexed in the transitive order) on x is maximal with respect to the lexicographic order.

Suppose for contradiction that x cannot be added to A so that their union is transitive, then the word of A on x contains 01, thus there are two vertices a and b, consecutive in A, such that $x \rightarrow a$, $a \rightarrow b$, and $b \rightarrow x$.



As A is minimal in size, we cannot add b to B, thus there exist c and d consecutive in B such that $b \to c$, $c \to d$, and $d \to b$. If $c \to a$, then we could reverse (a, b, c) keeping A and B transitive and increasing the dominancy word of A on x. Hence by our assumption that this word is maximal, we have $a \to c$. But now we reverse the cycle (x, a, c, d, b) and the dominancy word of A on x is increased. Contradiction. □

We note that Theorem 2 can be derived easily from the following result of Ryser [14] (Theorem 5.2) (see also Fulkerson [7] and Moon [13]). The cited proofs of this are more difficult than our cycle reversal argument. We need one more definition before we can state his theorem. If T is a tournament on vertex set $[n]$ with score sequence (in the increasing order) $s_1 \leq s_2 \leq \cdots \leq s_n$, then an *upset* is an edge $i \to j$ for $i < j$. We now state Ryser's result which says that the trivial lower bound on the number of upsets in a tournament with given score sequence can always be achieved by some tournament.

Theorem 3 *For every score sequence $s_1 \leq s_2 \leq \cdots \leq s_n$, there exists a tournament T on vertex set $[n]$ such that the number of upsets is*

$$\sum_{i\,:\,s_i > i-1} s_i - i + 1.$$

Theorem 3 implies that every vertex i satisfies at least one of the following properties:
- For every $j < i$, we have $i \to j$ in T.
- For every $j > i$, we have $j \to i$ in T.

Thus we can partition the vertices into two sets depending on which property they satisfy (vertices satisfying both properties are placed arbitrarily). This gives a 2-partition of the vertex set required for Theorem 2.

We next show that the 2-partition can be done in a balanced way which leads to a canonical construction of a 2-partition-transitive tournament. It was suggested by Rigollet and Thomassé that, given a score sequence in decreasing order, one can construct a tournament such that the restrictions on both odd and even indexed vertices are transitive and oriented in the indexed order. In the next section, we prove that this is possible.

3 Balanced partition into two transitive tournaments

We wish to prove the above-mentioned theorem by induction. The problem is that removing a vertex does not keep the score sequence in the right order, and for this reason, it is advantageous to make the statement stronger so that induction will be easier. We will prove the following statement concerning subsets of T.

Theorem 4 *Let T be a tournament and S any subset $\{x_1, x_2, \ldots, x_n\}$ of T with points labeled so that $d(x_i) \geq d(x_j)$ whenever $1 \leq i < j \leq n$. Then T can be transformed by a sequence of cycle reversals into a tournament T' having the property that the restriction of T' to both $\{x_1, x_3, x_5, \ldots\}$ and $\{x_2, x_4, x_6, \ldots\}$ is transitive and oriented in the indexed order.*

We first need some tools which link cycle reversal and degree. Here is one fundamental observation:

Lemma 1 *Let A be a subset of T such that $x \in A$ and $y \in A$. If $d(x) \geq d(y)$ and $d_A(y) - d_A(x) \geq p$, then there exist at least p vertices $z_1, z_2, \ldots, z_p$ in $T \setminus A$ such that $x \to z_i$ and $z_i \to y$ for any $1 \leq i \leq p$.*

Proof. Let $B = T \setminus A$, we have $d_A(x) + d_B(x) \geq d_A(y) + d_B(y)$. Since $d_A(y) - d_A(x) \geq p$, we get $d_B(x) - d_B(y) \geq p$. So there exist at least p elements of B satisfying the statement. $\Box$

Corollary 1 *Let x and y be two vertices of T such that $d(x) \geq d(y)$ and $y \to x$. Then there is a vertex z of T such that $x \to z$ and $z \to y$.*

Proof of Theorem 4. We will prove the theorem by induction. For $n = 2$, the theorem is clearly true.

We suppose it is true for n, and prove it for $n+1$, let $S = \{x_1, x_2, \ldots, x_{n+1}\}$ be a subset of T such that for any $1 \leq i < j \leq n+1$, $d(x_i) \geq d(x_j)$. In the following, when we refer to the index of an element, it is always with respect to its position in S. We apply the inductive hypothesis to the set $\{x_2, x_3, \ldots, x_{n+1}\}$ so that we now have two transitive tournaments $E = \{x_2, x_4, x_6, \ldots\}$ and $O = \{x_3, x_5, x_7, \ldots\}$ oriented in the indexed order. Let R be the set $T \setminus S$. Thus $\{R, O, E, \{x_1\}\}$ is a partition of T.

Our goal is to reverse cycles to add x_1 to O in such a way that it dominates all elements of O. The cycle reversals performed will leave E and O unchanged, and in this way we achieve our goal.

We assume that this is not possible, and among all possible ways of reversing cycles of T to make E and O transitive and oriented in the index order, we consider one that satisfies the following two conditions:

Condition 1 The number of elements of O which x_1 dominates is maximal, (in other words $d_O(x_1)$ is maximal).

Condition 2 The first element from O which dominates x_1 has minimal index.

We now analyze this tournament, determine properties of it, and then arrive at a contradiction. First we notice that there is an element in O that dominates x_1. For if there were not, we could add x_1 to O, achieving the claim of the theorem.

Proposition 1 *The point x_1 dominates x_3.*

Proof. If rather $x_3 \to x_1$, then by Corollary 1 (here $d(x_1) \geq d(x_3)$) there must be an element z such that $x_1 \to z$ and $z \to x_3$. This element z cannot belong to O because it dominates x_3. Reversing the cycle (x_1, z, x_3) leads to a contradiction of Condition 1. $\Box$

Now we denote by v the element of O with lowest index such that $v \to x_1$. As v is different from x_3, let u be the predecessor of v in O; let p be the number for which $v = x_{2p+1}$ and $u = x_{2p-1}$. Now we call O_1 the set of predecessors of u in O and O_2 the set of successors of v in O. Clearly $\{O_1, \{u, v\}, O_2\}$ is a partition of O. Moreover any element of O_1 is dominated by x_1 by the definition of v. Remark that there is no $z \in T$ such that $u \to z$ and $z \to v$. (If there were, then such a z could not belong to O and we could reverse the cycle (x_1, u, z, v), contradicting Condition 2.)

Proposition 2 *There is no $z \in T \setminus \{x_1\}$ such that $v \to z$ and $z \to u$.*

Proof. If there were, then we consider $A = \{u, v, x_1, z\}$. We have $d(u) \geq d(v)$ and $d_A(v) - d_A(u) = 1$, thus by Lemma 1 there would be an element w such that $u \to w$ and $w \to v$, contradicting the preceding remark. $\Box$

Now, except for x_1, the vertices u and v dominate the same set of vertices (Remark: $\{u, v\}$ is an interval (or autonomous subset) of $T \setminus \{x_1\}$). We discuss now which vertices in E are dominated by u and v: let E_1 be the set of vertices of E which dominate u and v. Conversely, let E_2 be the set of vertices of E dominated by u and v (see the figure at the end of the proof).

Proposition 3 *The set E_1 is not empty.*

Proof. Assume it were empty. Let $w = x_{2p}$ (we recall that $u = x_{2p-1}$ and $v = x_{2p+1}$). By the labeling, we know that $d(w) \geq d(v)$. Let $A = \{x_1, x_2, \ldots, x_{2p+1}\}$. We know that $d_A(v) = p+1$, as v dominates x_1 and $E_2 \cap A$, and $d_A(w) \leq p-1$ as w can only dominate the points in $\{x_1, x_3, \ldots, x_{2p-3}\}$. Thus $d_A(w) < d_A(v)$ and $d(w) \geq d(v)$, and by Lemma 1 there exists $z \in T \setminus A$ such that $w \to z$ and $z \to v$. We remark that z can be neither in $O \setminus A$ nor in $E \setminus A$ as it dominates v, thus it is in R (we recall that $\{R, O, E, \{x_1\}\}$ is a partition of T). Now, reversing the cycle (x_1, u, w, z, v) leads to a contradiction of Condition 2. $\square$

Proposition 4 *The vertex x_1 is dominated by every element of E_1. Moreover, every element of E_1 dominates every element of O_1.*

Proof. Suppose for contradiction that there is an element $b \in E_1$ such that $x_1 \to b$. Then reversing the cycle (x_1, b, v) leads to a contradiction of Condition 1. Suppose for contradiction that there are elements $a \in O_1$ and $b \in E_1$ such that $a \to b$. Then reversing the cycle (x_1, a, b, v) leads to a contradiction of Condition 2. $\square$

Proposition 5 *The set E_1 is an initial section of E and E_2 is a final section of E.*

Proof. Suppose for contradiction that there are elements $x_{2i} \in E_2$ and $x_{2i+2} \in E_1$. We consider $A = \{u, v, x_{2i}, x_{2i+2}\}$. We have $d(x_{2i}) \geq d(x_{2i+2})$ and $d_A(x_{2i+2}) - d_A(x_{2i}) = 1$, thus by Lemma 1 there is an element z such that $x_{2i} \to z$ and $z \to x_{2i+2}$. Following Proposition 4, the element z cannot be equal to x_1. The element z cannot belong to E, then reversing the cycle $(x_{2i}, z, x_{2i+2}, v, x_1, u)$ gives a contradiction of Condition 2. $\square$

Let $q = |E_1| > 0$ and in particular, we know that $x_2 \in E_1$. Now, let $A = \{x_1, u, v\} \cup E_1 \cup O_1$, we know that $d_A(x_2) = p+q$ as it dominates any element of A. Moreover $d_A(x_1) = p-1$, then $d_A(x_2) - d_A(x_1) = q+1$. So, as $d(x_1) \geq d(x_2)$, we apply Lemma 1 in order to find a set $W = \{w_1, w_2, \ldots, w_{q+1}\}$ such that $w_i \to x_2$, $x_1 \to w_i$ and $w_i \in T \setminus A$ for any i. Clearly, as x_2 dominates all the elements of E, the set $E \cap W$ is empty. Note that the set $W \cap R$ is empty. Indeed, if there were a $w_i \in W \cap R$; reversing the cycle (w_i, x_2, v, x_1) would lead to a contradiction of Condition 1. All the elements of W are therefore in O_2. We assume that the set $\{w_1, w_2, \ldots, w_{q+1}\}$ is indexed in the transitive order. The element w_{q+1} (with maximal index among the elements of W) is essential in the final contradiction.

Proposition 6 *Every element of W dominates every element of E_2.*

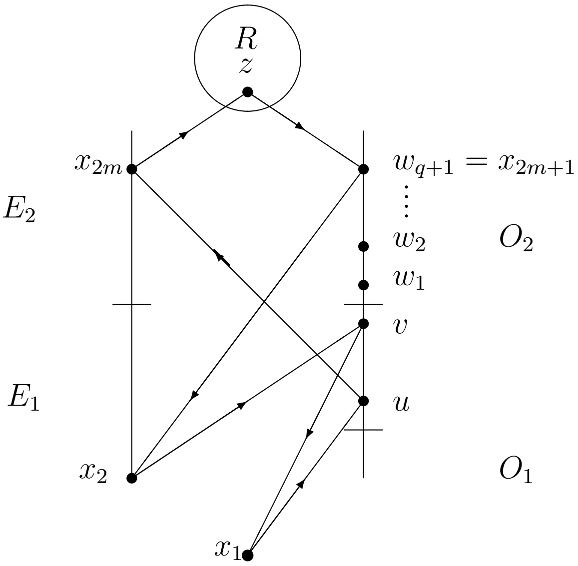
Proof. Assume there were $w \in W$ and $a \in E_2$ such that $a \to w$. We could then reverse the cycle (w, x_2, v, x_1, u, a) which would lead to a contradiction of Condition 2. $\square$

Now, as w_{q+1} is in O, there exist an m such that $w_{q+1} = x_{2m+1} \in O$. The element $x_{2m} \in E$ plays now the fundamental role in the contradiction.

Proposition 7 *The element x_{2m} belongs to E_2.*

Proof. It suffices to remark that $w_{q+1} = x_{2m+1}$ has at least q elements preceding it in O with respect to the index, so the element x_{2m} also has q elements preceding it in E with respect to the index. The number of elements in E_1 is q and E_1 is an initial section of E, hence $x_{2m} \in E_2$. $\square$

Now, consider the set $A = \{x_1, x_2, \ldots, x_{2m+1}\}$. We will compare the degrees of $w_{q+1} = x_{2m+1}$ and x_{2m} within this set. We essentially have the following picture:



The degree of $w_{q+1} = x_{2m+1}$ in A is at least $|E_2 \cap A|$, as w_{q+1} dominates every element of E_2. Therefore $d_A(w_{q+1}) \geq m - q$.

The degree of x_{2m} in A is at most $2m - [(q+1) + (m-1) + 2]$, as any element of W dominates x_{2m}; all $m-1$ first elements of E dominate x_{2m}; and both u and v dominate x_{2m} (u and v dominate all elements of E_2). So $d_A(x_{2m}) \leq m - q - 2$.

As $d(x_{2m}) \geq d(x_{2m+1})$ and $d_A(x_{2m}) < d_A(x_{2m+1})$, we may now apply Lemma 1 to show that there is an element $z \in T \setminus A$ such that $x_{2m} \to z$ and $z \to x_{2m+1}$. We remark that z cannot belong to $E \setminus A$ because x_{2m+1} dominates any element of E_2, moreover, z does not belong to $O \setminus A$ because x_{2m+1} dominates any element of O_2. Thus $z \in R$.

We now consider the following cycle on seven elements (picture):

$$(z, x_{2m+1}, x_2, v, x_1, u, x_{2m}).$$

Reversing this cycle leads to a contradiction of Condition 2. This completes the proof of the theorem. □

4 Consequences of balanced 2-partitions

In this section we will give two applications of Theorem 4. A digraph D_n is called *unavoidable* if it is a subdigraph of every tournament T on n vertices. There is extensive literature on unavoidable digraphs (cf. e.g. [11] and the literature cited there). In [11] they determine the asymptotics for the maximum number of edges in an unavoidable digraph D_n. In a natural way, we define a digraph D_n on n vertices to be *score unavoidable* if for every n component score sequence, there exists a tournament T with the given score sequence and containing D_n as a subdigraph. We prove the following similar result.

Theorem 5 *Let D_n be a score unavoidable digraph with the maximum number of edges. Then*

$$\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \leq |E(D_n)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The lower bound is an immediate consequence of Theorem 4.

To prove the upper bound, we will show that if D_n is a score unavoidable digraph with outdegrees $d_1 \leq d_2 \leq \cdots \leq d_n$, then

$$d_k \leq \left\lceil \frac{k-1}{2} \right\rceil, \qquad 1 \leq k \leq n.$$

The upper bound then follows trivially from this. The claim follows from the observation that if a tournament T has k vertices with score at most s_k, then D_n must have at least k vertices with outdegree not greater than k; this is equivalent to saying that $d_k \leq s_k$ where the s_i's and the d_i's are both in nondecreasing order. Hence, we only need to give for each k, a tournament score sequence on n vertices with $s_k = \lceil (k-1)/2 \rceil$ (where s_k is the k-th smallest score). We take the score sequence of a tournament which is as regular as possible on the first k vertices, and for the larger indexed vertices i, i beats j if $i > j$. More precisely, this score sequence is given by $s_i = \lfloor (k-1)/2 \rfloor$ for $i \leq k/2$, $s_i = \lceil (k-1)/2 \rceil$ for $k/2 < i \leq k$, and $s_i = i - 1$ for $k < i \leq n$. This completes the proof. □

The next goal of this section is to use Theorem 4 in order to give an algorithm to construct a tournament from a given score sequence. The advantage of using this theorem is that we can specify roughly half of the edges at the beginning.

A *bipartite tournament* is an orientation of a complete bipartite graph. The score sequence of a bipartite tournament consists of two sequences of integers, these are the outdegrees of the vertices of each class of the bipartite tournament.

First we will recall the characterization of tournament score sequences and bipartite tournament score sequences. We remark that all of these are special cases of the characterization by Moon [12] for n-partite tournaments. We also mention that Bang and Sharp [2] give a very nice proof of the first part of the following theorem using Hall's theorem which can easily be extended to the n-partite case.

Theorem 6 (Landau [10]; Gale [8] and Ryser [14])
(1) *A sequence of integers $S = (s_1, \ldots, s_n)$ is a score sequence of a tournament if and only if for any subset A of $[n]$, we have*

$$\sum_{i \in A} s_i \geq \binom{|A|}{2},$$

with equality when $|A| = n$.
(2) *Two sequences of integers $S = (s_1, \ldots, s_n)$ and $T = (t_1, \ldots, t_m)$ are the score sequence of a bipartite tournament if and only if for any $A \subset [n]$ and $B \subset [m]$, we have*

$$\sum_{i \in A} s_i + \sum_{j \in B} t_j \geq |A| \cdot |B|,$$

with equality when $|A| = n$ and $|B| = m$.

As a consequence of Theorem 4, if $S = (s_1, s_2, \ldots, s_n)$ is a score sequence of tournament given in the increasing order, we know that we can find a tournament having this score sequence such that the restrictions on both evens and odds are transitive tournaments oriented from larger indexed vertices to smaller indexed vertices. It follows that

Corollary 2 *If $s_1 \leq s_2 \leq \cdots \leq s_n$ is the score sequence of a tournament given in the increasing order, then $\{s_1, s_3 - 1, s_5 - 2, s_7 - 3, \ldots\}$ and $\{s_2, s_4 - 1, s_6 - 2, s_8 - 3, \ldots\}$ are score sequences of a bipartite tournament.*

This remark gives rise to the following algorithm to construct a 2-partition-transitive tournament from a given score sequence S:

1) Order $S = \{s_1, s_2, s_3, \ldots, s_n\}$ in an increasing way.

2) Split it into

$$S_1 = \{s_1, s_3 - 1, s_5 - 2, \ldots\} \quad \text{and} \quad S_2 = \{s_2, s_4 - 1, s_6 - 2, \ldots\}.$$

We now apply the algorithm (Beineke and Moon [3] pp. 60–61) to construct a bipartite tournament from a bipartite score sequence.

3) Order S_2 in the increasing order, let $S_2 = \{j_1, j_2, \ldots, j_t\}$.

4) As long as S_1 is not empty, pick a minimal integer i in S_1.

5) Let k be the minimal index such that $j_k = j_i$ and l be the maximal index such that $j_l = j_i$.

6) Add the arrows $j_s \to i$ and decrease j_s by 1 whenever $s > l$ or $k \leq s \leq l + k - i$. In the other cases, add the arrow $i \to j_s$.

7) Delete i from S_1 and apply 5)

This algorithm is twice as fast as the usual algorithm used to construct tournaments from score sequences (see [13], p. 73).

5 Number of triangle reversals

As our proofs give an algorithm to construct a 2-partition-transitive tournament via cycle (triangle) reversals, it is natural to ask how many triangle reversals are needed. This question is related to work done by Brualdi and Qiao [5], where they investigate the interchange graph for a given score sequence. Let S be a score sequence of length n and let $\tilde{G}(S)$ be the graph whose vertices are all labeled tournaments with score sequence S, and two tournaments T and T' are connected by an edge if T can be transformed into T' via a triangle reversal. Brualdi and Qiao give an example of two tournaments whose distance is $(n-1)^2/4$ (their Corollary 3.9). The difference with our approach is that we are not concerned with labelings. For our purposes, we are interested in the graph $G(S)$ whose vertices are (unlabeled) tournaments with a given score sequence, hence our graph is a contraction of their graph for all isomorphism classes. In the example in [5], the two tournaments at large distance are isomorphic, and hence the same vertex in our graph. The results of Brualdi and Qiao give an upper bound for the diameter of the graph $G(S)$. The next theorem is not explicitly stated in [5], but follows from their techniques.

Theorem 7 *For any score sequence S, the diameter of the graph $G(S)$ is at most $(n-1)(n-2)/2$.*

Proof. Let T and T' be two tournaments with identical score sequence S. Consider the simple graph $T - T'$, with the same vertex set and whose edges are those directed edges from T that are reversed in T'. The first observation is that $T - T'$ is an Euler graph (in the sense that for every vertex, the in-degree equals the out-degree — it may not be connected). Thus this can be partitioned into edge-disjoint cycles, and we have a set of cycles whose reversals gets us from T to T'. At most all of the edges can be contained in this graph, hence at most $\binom{n}{2}$ edges. The second observation is that in a tournament, the reversal of a cycle of length k can be accomplished by $k - 2$ triangle reversals (this follows easily by induction).

Let $C_1, C_2, \ldots, C_N$ be the cycles, $c_i = |C_i|$. Thus we want to bound $\sum_{i=1}^{N}(c_i - 2)$. Let $t = \sum_{i-1}^{N} c_i$. We have

$$t \leq \binom{n}{2} \qquad \text{and} \qquad t \leq nN.$$

The second inequality implies that $N \geq t/n$ and it follows that the number of triangle reversals is:

$$t - 2N \leq t - 2t/n \leq \frac{(n-1)(n-2)}{2}$$

. □

One approach to get a better bound for the diameter of $G(S)$ is given in the following problem. Let $\rho(D)$ denote the maximum number of directed cycles in an Euler graph D.

Problem 1 *Find the best* c *such that*

$$\min_{|V(D)|=n} \rho(D) \leq cn^2.$$

We show that there are tournaments which are at a distance cn^2 from a 2-transitive-partition tournament with the same score sequence. We consider the Paley tournament (also called the quadratic residue tournament). Let $p \equiv 3 \pmod 4$ be a prime, and consider the tournament defined on the finite field of p elements, i.e. $[p-1]$, given by

$$i \to j \quad \text{if and only if} \quad \left(\frac{i-j}{p}\right),$$

where $\left(\frac{a}{p}\right)$ is the quadratic residue of a modulo p, defined to be 0 if a is zero, 1 if a is a square modulo p, and -1 if a is not a square. This is a well-defined tournament as -1 is not a square for such p and hence exactly one of $i-j$ and $j-i$ is a square. This tournament has many interesting properties and has been studied frequently (cf. e.g. [1], [4], [9], [13]).

We prove the following result:

Theorem 8 *For the Paley tournament on* p *vertices, at least* $(1-o(1))p^2/32$ *triangle reversals are required to transform it into a balanced* 2*-partition transitive tournament.*

The one important property we will use is that for any two vertices i and j, the number of vertices dominated by both i and j is $(p-3)/4$, independent of which two vertices we consider. As the proof is rather short, we include it. The number of vertices d dominated by both i and j is given by the sum (over all elements of the field of p elements not equal to i and j)

$$\begin{aligned} d &= \frac{1}{4}\sum_{k\neq i,j}\left(\left(\frac{i-k}{p}\right)+1\right)\left(\left(\frac{j-k}{p}\right)+1\right) \\ &= \frac{1}{4}\sum_{k\neq i,j}\left(\frac{i-k}{p}\right)\left(\frac{j-k}{p}\right)+\frac{1}{4}\sum_{k}\left(\frac{i-k}{p}\right)+\frac{1}{4}\sum_{k}\left(\frac{j-k}{p}\right)+\frac{p-2}{4} \end{aligned}$$

We now use the fact that $\sum_k \left(\frac{k}{p}\right) = 0$, and that the quadratic character is multiplicative.

$$\begin{aligned} d &= \frac{1}{4}\sum_{k\neq i,j}\left(\frac{(i-k)/(j-k)}{p}\right)+\frac{p-2}{4} \\ &= \frac{1}{4}\sum_{k\neq i,j}\left(\frac{1+(i-j)/(j-k)}{p}\right)+\frac{p-2}{4} \end{aligned}$$

We notice that the sum is over all numbers in the field except 0 and 1. Hence its sum is -1 and $d = (p-3)/4$.

Proof of Theorem 8. We will get a lower bound on the number of edge reversals needed, and in this way we get a lower bound on the number of triangle reversals as well.

Let A and B be the final balanced partition, where $A = \{a_1, \ldots, a_{(p-1)/2}\}$ and $B = \{b_1, \ldots, b_{(p+1)/2}\}$, in the transitive order. Consider the edges from the pair of vertices a_{2i-1} and a_{2i} to all larger indexed vertices in A. There can be at most $(p-3)/4$ edges correct, so we need to reverse at least $(p-1)/2 - (p-3)/4 - 2i = (p+1)/4 - 2i$ edges for each $1 \leq i \leq (p-1)/8$. Adding these up we get:

$$\left(\frac{p+1}{4} - 2\right) + \left(\frac{p+1}{4} - 4\right) + \left(\frac{p+1}{4} - 6\right) + \cdots \geq \frac{p^2 - 14p + 33}{64}.$$

Each of these edges must be reversed, and similarly we need at least that many in B, hence at least $(1 - o(1))p^2/32$ edge reversals in the tournament. Any triangle can correct at most one edge. If all three vertices are in set A (equivalently in B), then it can correct two, but at the cost of making what was a good edge bad. If the vertices are in both A and B, then only one edge is switched. Hence we need at least $(1 - o(1))p^2/32$ triangle reversals in the Paley tournament. □

A natural question is that of how fast one can find a sequence of triangle reversals to get from an arbitrary tournament T to either a given tournament T' with equal score sequence, or to a balanced 2-partition-transitive tournament with equal score sequence. We will show that both of these can be done in $O(n^2)$.

Theorem 9 *Given two tournaments T and T' with the same score sequence, it is possible to find a sequence of triangle reversals in $O(n^2)$ time that converts T into T'.*

As a simple consequence of this theorem and the above algorithm that constructs a balanced 2-partition-transitive tournament in $O(n^2)$ time, we have the following corollary.

Corollary 3 *Given a tournament T, it is possible to find a sequence of triangle reversals in $O(n^2)$ time that convert T into a balanced 2-partition-transitive tournament.*

Proof of Theorem 9 We consider the directed graph $D = T - T'$ mentioned above where the edges are those of T that are reversed in T'. As we mentioned before, this graph is Euler (and can be found trivially in $O(n^2)$ time).

We can find the components of D in $O(E(D))$ time and then for each (Eulerian) component D_i of D we can apply any algorithm which finds an Euler circuit in D_i in $O(E(D_i))$ time (cf. [6]). After this we can consider each D_i as an Euler circuit as well.

We then consider the components one at a time. In a component, choose a starting vertex u, called a *pivot* vertex, of the Euler circuit and in the list describing the Euler curcuit we check the direction of the edge of T from the given vertex v to the pivot vertex u. Then starting from u, we divide the Euler circuit into triangles. Every triangle will contain u and two adjacent vertices in the Euler circuit. We do this recursively by decreasing the length of the Euler circuit one edge at a time.

To state it more precisely, let C be the current Euler circuit of T, starting and ending at u with edge sequence starting with e, f, g . We shall use v for the third vertex of C, i.e. for the endpoint of f. Our algorithm is to apply the following recursive procedure repeatedly for each i with initial Euler circuit $C = D_i$.

If u dominates v (in T) then we apply the algorithm recursively to the Euler circuit of T we get from the current one by replacing e, f with (u, v) and then reverse the triangle $e, f, (v, u)$. Otherwise we reverse the triangle first and we apply the algorithm recursively as before, unless when $g = (v, u)$, in that case we apply the algorithm recursively to the Euler circuit we get from the current one by removing e, f, g . This may lead to the empty Euler circuit, in which case the algorithm stops.

It is easy to see that the algorithm reverses only triangles with all three edges in T and after its termination precisely the edges of D are reversed. Therefore the tournament T' is obtained.

Clearly at most $O(E(D_i))$ edges have to be checked in component i. These are additive and therefore $O(n^2) + O(E(D)) = O(n^2)$ time (and space) is enough to find the triangle reversals. □

We end with the following natural question related to Problem 1.

Problem 2 Determine the maximum diameter of the graph $G(S)$ over all score sequences S.

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