Chapter 4

Analytic equations

In the preceding chapter we allowed the dependent variable (y or u) to be complex, but required the independent variable (x or t) to be real. We found that the solutions could be constructed in terms of elementary functions. In the present chapter, not only the dependent variable but also the independent variable is allowed to be complex. The derivatives are therefore complex derivatives, so the functions differentiated must be *analytic*. We shall again be able to construct the solutions, although now they will take the form of power-series, and may represent new, unfamiliar functions.

We provide in §4.1 statements, without proof, of the few facts regarding analytic functions that we require in the present chapter. However, in this and the next chapter, some familiarity with methods of complex analysis, as described in books on complex-function theory, is desirable.

4.1 Power Series

Consider the power series

$$\sum_{k=0}^{\infty} a_k z^k \tag{4.1}$$

where the coefficients $\{a_k\}$ are prescribed complex numbers and z is a complex variable. Any such power series has a *radius of convergence* R. R may be zero, in which case the series fails to converge except in the uninteresting case when $z = 0$. If R is not zero, then the series converges for each z with $|z| < R$ and diverges for any z with $|z| > R$. It is also possible to have $R = \infty$, meaning that the series converges for all z in the complex plane. The reasoning leading to these conclusions is the following.

Suppose that the series (4.1) converges for some value of z, say $z = z_0$. The convergence of $\sum a_k z_0^k$ implies that $\left|a_k z_0^k\right| \to 0$ as $k \to \infty$ and therefore that $|a_k| < M |z_0|^{-k}$ for some sufficiently large M. Therefore if $|z| < |z_0|$ then $\left|a_k z^k\right| \leq M\rho^k$ with $\rho = |z/z_0| < 1$. The series $\sum a_k z^k$ is therefore absolutely convergent, by comparison with the convergent series $M \sum \rho^k$. The radius of convergence R is now the largest number r such that $\sum a_k z^k$ converges if $|z| < r$. The series (4.1) therefore represents a complex function within the disk of radius R, say

$$a(z) = \sum_{k=0}^{\infty} a_k z^k.$$

What is more, the function a is analytic for $|z| < R$, and its derivative is obtained by term-by-term differentiation of the series:

$$a'(z) = \sum_{k=0}^{\infty} k a_k z^{k-1},$$

the radius of convergence of the series on the right-hand side of preceding equation being the same, R, as that of the series for a.

There is a sometimes convenient formula for the radius of convergence of the series (4.1):

$$R^{-1} = \lim\sup |a_n|^{1/n}. \tag{4.2}$$

The feature of analytic functions that is used in the present chapter is that any analytic function can be represented by a power-series expansion. More precisely, suppose a is an analytic function of the complex variable $z = x + iy$ in a domain D of the complex plane, and let z_1 be any point of D. Then

$$a(z) = \sum_{k=0}^{\infty} a_k (z - z_1)^k, \tag{4.3}$$

where the series has a nonzero radius of convergence R, i.e., converges for $|z - z_1| < R$. The radius of convergence R is the distance from z_1 to the boundary of D.

4.2 Linear, analytic equations

A linear, analytic equation is one for which the coefficient functions are analytic and therefore possess convergent power-series expansions, as in equation (4.3) above. The simple conclusion will be that the solutions also possess convergent power-series expansions. This provides a powerful method for constructing solutions. Even if the coefficients are real and we are ultimately interested in real solutions of these equations, it is useful and illuminating to allow for solutions in the form of complex functions of a complex independent variable. To emphasize this we employ standard, complex notation z for the independent variable and w for the dependent variable, and consider the differential equation

$$Lw \equiv w^{(n)} + a_1(z)\, w^{(n-1)} + \cdots + a_{n-1}w' + a_n w = 0. \tag{4.4}$$

We have written the homogeneous equation but, as usual, we shall also be interested in solutions of the inhomogeneous equation.

The definition of linear dependence and independence is the same as it was for real equations, with only the obvious changes: we need to allow complex coefficients, and we need to consider the linear dependence or independence, not on an interval, but on a domain in the complex plane:

Definition 4.2.1 *The functions $w_1, w_2, \ldots, w_m$ are linearly dependent in the domain D of the complex plane if there exist complex constants $c_1, c_2, \ldots, c_m$, not all zero, such that*

$$c_1 w_1(z) + c_2 w_2(z) + \cdots + c_m w_m(z) = 0$$

at each point z of D.

As in the case of real equations, we shall find that any solution of equation (4.4) is a linear combination of a basis of n linearly independent solutions.

By a shift of the independent variable ($z' = z - z_1$) we may assume without loss of generality that the expansions of the coefficients may be written

$$a_j(z) = \sum_{k=0}^{\infty} a_{jk} z^k, \quad j = 1, 2, \ldots, n \tag{4.5}$$

where each of the n series converges if $|z| < r$. Normally we take for r the smallest of the radii of convergence of the coefficient functions.

The important features of the theory are already in evidence if $n = 2$, so we now turn to this.

4.3 Power-series expansions for $n = 2$

The basic equation (4.4) can be written

$$w'' + p(z)\, w' + q(z)\, w = 0 \tag{4.6}$$

where p and q possess power-series expansions like that of equation (4.5), with coefficients $\{p_k\}$ and $\{q_k\}$ respectively. If there is *any* solution w which has a derivative in a neighborhood of the origin, i.e., is analytic there, it too has a power-series expansion

$$w(z) = \sum_{k=0}^{\infty} w_k z^k. \tag{4.7}$$

Example 4.3.1 Consider equation $w'' - w = 0$ and further impose the initial conditions $w(0) = 1$, $w'(0) = 1$. In the expansion (4.7) we must then have $w_0 = 1$, $w_1 = 1$. Substitution of that expansion in the differential equation then gives the *recursion formula* $w_{k+2} = w_k/(k+1)(k+2)$. It is easy to check that $w_k = 1/k!$: it's true for $k = 0, 1$ and follows, by induction, from the recursion formula for the general case. Thus $w(z) = \sum_{k=0}^{\infty} z^k/k!$, which we recognize as $\exp z$. $\square$

In the preceding example, the coefficients were $p = 0$ and $q = 1$; these are analytic in the entire plane, and we find that the solutions are likewise analytic in the entire plane. In the general case when the coefficients have power-series expansions convergent with some radius of convergence R (which could be infinite, as in this example), the procedure is to write a power-series expansion for w with coefficients to be determined, and substitute this series into equation (4.4). This requires the multiplication of two power series to obtain the product power series. The formula for such a product is

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{4.8}$$

Suppose now we assume provisionally that the equation (4.4) possesses a power-series solution $w = \sum w_k z^k$. Putting off the question whether we can expect it to converge, we proceed formally and determine the coefficients $\{w_k\}$, as follows. Substitution of the power-series expression into equation (4.4) results in the formula

$$\sum_{k=0}^{\infty} k(k-1) w_k z^{k-2} + \sum_{k=0}^{\infty} k w_k z^{k-1} \sum_{j=0}^{\infty} p_j z^j + \sum_{k=0}^{\infty} w_k z^k \sum_{j=0}^{\infty} q_j z^j = 0.$$

By collecting coefficients of like powers of z we obtain, on the left-hand side of the preceding equation, the following power series:

$$\sum_{k=0}^{\infty} \left\{ (k+1)(k+2) w_{k+2} + \sum_{l=0}^{k} p_{k-l}(l+1) w_{l+1} + \sum_{l=0}^{k} q_{k-l} w_l \right\} z^k = 0.$$

If this is to hold in any neighborhood of the origin, it must be that each coefficient vanishes separately. This gives the *recursion formula*

$$(k+1)(k+2)\,w_{k+2} + \sum_{l=0}^{k} p_{k-l}\,(l+1)\,w_{l+1} + \sum_{l=0}^{k} q_{k-l}w_l = 0, \tag{4.9}$$

for $k = 0, 1, \ldots$. If w_0 and w_1 are given, this determines all subsequent coefficients. In particular, if $w_0 = w_1 = 0$, then all subsequent cofficients also vanish. Therefore if the solution with vanishing initial data is analytic in a domain containing the origin, it vanishes identically there, by a standard theorem in complex-function theory. From this we can infer, as in Chapter 2, that if w_1 and w_2 are linearly independent solutions in D, then any solution is a linear combination of them. More generally, since w_0 and w_1 represent the values of w and its first derivative at $z = 0$, this recursion formula shows that the solution is uniquely determined by its initial values, at least formally.

The Wronskian plays a similar role for solutions of the homogeneous equations in the present, complex case as it did in the previous, real case. However, for more general functions, not necessarily satisfying such an equation, it plays a somewhat different role. For example,

Theorem 4.3.1 *If u and v are analytic functions in a domain D of the complex plane, and if their Wronskian vanishes identically on D, then they are linearly dependent on D.*

Proof: Assume that v does not vanish identically on D or the conclusion is trivially true. Let $v(z_0) \neq 0$ for some $z_0 \in D$; then v does not vanish on a neighborhood $d \subset D$. We may therefore write the equation expressing the vanishing of their Wronskian as

$$u' - \frac{v'}{v}u = 0, \ z \in d.$$

Since $u(z) = v(z)$ is a solution, $u(z) = cv(z)$ is the unique solution of this equation on d for some choice of c. Thus the function $u(z) - cv(z)$ is analytic on D and vanishes on an open subset of D, and must, by a standard theorem in complex-function theory, therefore vanish on all of D. $\square$

This contrasts with the real case. For example, the functions

$$u(x) = x^3 \text{ and } v(x) = \begin{cases} x^3 \text{ if } x > 0 \\ -x^3 \text{ if } x \leq 0 \end{cases}$$

are linearly independent on the real line, but their Wronskian vanishes identically there.

To turn the formal solution into a genuine solution, we need to know that the series converge.

Example 4.3.2 Consider the equation

$$w'' - \frac{4}{1-z}w' + \frac{2}{(1-z)^2}w = 0.$$

The coefficients p and q both have power-series expansions with radius of convergence $R = 1$. The equation has a pair of linearly independent solutions $w_1 = (1-z)^{-1}$ and $w_2 = (1-z)^{-2}$. These also have power-series expansions with the same radius of convergence, $R = 1$. □

Example 4.3.3 Consider the equation

$$w'' - \frac{2}{(1+z)^2}w = 0.$$

Here $p = 0$ and q has a power series with radius of convergence $R = 1$. This equation has the linearly independent pair of solutions $w_1 = (1+z)^2$ and $w_2 = (1+z)^{-1}$. The latter has a power series expansion with $R = 1$ but the former has a power series with infinite radius of convergence. □

Example 4.3.4 Now consider the equation

$$w'' + \frac{1}{1-z}w' = 0.$$

It has solutions $w_1 = 1$ and $w_2 = 1 - 2z + z^2$. Here the coefficient q has a power series with radius of convergence $R = 1$, and both solutions have power series with infinite radius of convergence. □

In the preceding examples the coefficients have power series convergent for $|z| < 1$. It is possible, as we see from these examples, that this gives the maximal disk of convergence for any power-series solution, or that there may be one or even two independent solutions that have power series convergent beyond this disk. We see in the next section that any power-series solution converges with at least the radius of convergence of coefficient functions.

4.4 The method of majorants

The method for establishing the convergence of the power series with coefficients given by the recursion formula (4.9) is remarkably simple. It is based on the idea of a *majorizing* series:

Definition 4.4.1 *The series* $\sum_{k=o}^{\infty} A_k z^k$ *majorizes the series* $\sum_{k=0}^{\infty} a_k z^k$ *if* $A_k \geq |a_k|$, $k = 0, 1, \ldots$.

It is seen from the definition that the coefficients of the majorizing series are real and non-negative. The basic fact about two power series related in this way is that if the majorizing series converges for $|z| < r$, so also does the other series. The proof of this remark may be based on the Cauchy convergence criterion: the infinite series of complex numbers $\sum c_n$ converges if and only if, given any positive number ϵ, there is an integer N such that

$$\left|\sum_{k=m}^{n} c_k\right| < \epsilon \text{ provided } n > m \geq N.$$

Suppose the majorizing series has radius of convergence R and let ρ be any real number in the interval $(0, R)$. The majorizing series then converges for any z such that $|z| \leq \rho$. For such z we have

$$|\sum_{n}^{m} a_k z^k| \leq \sum_{n}^{m} |a_k||z|^k \leq \sum_{n}^{m} A_k \rho^k.$$

Given $\epsilon > 0$ we can choose N so that, for values of n and m greater than N (and $n > m$), the last sum on the right is less than ϵ. Hence the radius of convergence of the series $\sum a_k z^k$ cannot be less than ρ and, since ρ is *any* positive number less than R, the radius of convergence cannot be less than R.

Consider now the differential equation (4.6) and suppose its coefficients p and q have power-series expansions

$$q(z) = \sum_{k=0}^{\infty} q_k z^k \text{ and } p(z) = \sum_{k=0}^{\infty} p_k z^k$$

which are majorized, respectively, by the power series

$$Q(z) = \sum_{k=0}^{\infty} Q_k z^k \text{ and } P(z) = \sum_{k=0}^{\infty} P_k z^k.$$

Denote by w_0 and w_1 the initial data for the solution w of equation (4.6).

Lemma 4.4.1 *Let* $W_0 \geq |w_0|$ *and* $W_1 \geq |w_1|$. *Then the solution* $W(z)$ *of the initial-value problem*

$$W'' = P(z) W' + Q(z) W, \quad W(0) = W_0, W'(0) = W_1$$

majorizes the solution $w(z)$ *of the initial-value problem consisting of equation (4.6) together with the initial data* w_0, w_1.

Proof: The recursion relation for the solution of the majorant initial-value problem is the same, except for a change of sign, as that given by equation (4.9) above. Comparison of the successive coefficients in the two cases, and a simple induction, provides the conclusion. $\square$

Suppose now that the lesser of the radii of convergence of the power series for p and q is R, and let r be any positive number with $r < R$. Then choose ρ so that $r < \rho < R$. There is some $M > 0$ such that the series $M \sum_{k=0}^{\infty} \rho^{-k} z^k$, convergent for $|z| < \rho$, majorizes each of the series for p and q. If the series with coefficients $M\rho^{-k}$ majorizes that for q, so also does the series with coefficients $M^2 (k+1) \rho^{-k}$ if $M \geq 1$; we are free to assume this. Then the majorizing functions are

$$P(z) = M \sum_0^{\infty} \frac{z^k}{\rho^k} = \frac{M}{1 - z/\rho}, \; Q(z) = M^2 \sum_0^{\infty} (k+1) \frac{z^k}{\rho^k} = \frac{M^2}{(1 - z/\rho)^2}.$$

The equation

$$W'' = \frac{M}{1 - z/\rho} W' + \frac{M^2}{(1 - z/\rho)^2} W$$

has the solution $W = A(1 - z/\rho)^{\alpha}$ where

$$\alpha = \left\{ 1 - \rho M - \sqrt{(1 - \rho M)^2 + 4\rho^2 M^2} \right\} < 0.$$

For this solution $W(0) = A$ and $W'(0) = -\alpha A/\rho$. If A is chosen large enough these are positive numbers dominating the initial data w_0 and w_1. The solution of the majorizing problem now is seen to have radius of convergence ρ. It follows that the power-series solution of the initial-value problem converges for $|z| = r$. But r was any number less than R, the least radius of convergence of the coefficient series for p and q. Therefore the power-series solution for w must also converge with radius of convergence at least equal to R. We summarize this.

Theorem 4.4.1 *Any power-series solution of equation (4.6) has radius of convergence at least equal to R, where R is the lesser of the radii of convergence of the power series for the two coefficient functions p and q.*

PROBLEM SET 4.4.1

1. Let $w' + p(z) w = 0$ be a first-order equation with $p(z) = \sum_{k=0}^{\infty} p_k z^k$ for $|z| < R$. Obtain the formal power-series solution of the equation.

2. Refer to Example 4.3.1 and impose instead the initial data $w(0) = 1$, $w'(0) = -1$. Identify the solution with a known, elementary function. The same with the initial data $w(0) = 1$, $w'(0) = 0$ and $w(0) = 0$, $w'(0) = 1$.

3. Prove the correctness of the relation (4.8).

4. Prove the statement in the text that the radius of convergence of the series $\sum a_k z^k$ is at least as great as that of a majorizing series $\sum A_k z^k$.

5. In Example 4.3.3 find the coefficients $\{q_k\}$ and write the recursion formula (4.9) explicitly for this example. Carry out the calculation of the coefficients $\{a_k\}$ of the expansion $w = \sum a_k z^k$ via this formula, for the initial data $w(0) = a_0 = 1$, $w'(0) = a_1 = 2$, and obtain the function $w_1(z)$ of Example 4.3.3 in this way.

6. Rewrite the equation of Example 4.3.3 as $(1+z)^2 w'' - 2w = 0$. Obtain a recursion formula by substituting the series for w in this equation instead. Solve the initial-value problem for the initial data given in the preceding problem using the recursion formula that you obtain this way.

7. Find the recursion formula for the coefficients of the power-series solution of the equation

$$w'' + \frac{1}{1+z^2} w = 0.$$

   What is the radius of convergence for general initial conditions?

8. Find a lower bound for radius of convergence for power-series solutions to the equation

$$\left(1 + Az + Bz^2\right) w'' + (C + Dz)\, w' + Ew = 0.$$

   Here $A, \ldots, E$ are real constants. Do *not* work out the recursion formula. *Do* rely on Theorem 4.4.1.

9. Put $A = C = 0$ in the preceding problem. Find a condition on B, D, E for this equation to possess a poynomial solution (a non-trivial condition, please: do not set w equal to zero).

The next three problems refer to Legendre's equation:

$$\frac{d}{dz}\left(\left(1 - z^2\right)\frac{dw}{dz}\right) + \lambda w = 0, \tag{4.10}$$

where λ is a constant.

10. Find the recursion formula for power-series solutions $\sum a_k z^k$ about the origin in the form

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k \tag{4.11}$$

and give conditions on a_0, a_1 (i) ensuring that the solution is an even function of z and (ii) that the solution is an odd function of z.

11. Find a condition on λ for Legendre's equation to have a polynomial solution.

12. Suppose, as in Problem 11, that one solution $u(z)$ of Legendre's equation is bounded at $z = 1$. Assuming $u(1) \neq 0$, show that the second solution $v(z)$ becomes infinite like $\ln(1 - z)$ there.

13. Airy's equation is

$$w'' + zw = 0.$$

Obtain the power-series expansion under the initial data $w(0) = 1, w'(0) = 0$. What is its radius of convergence?

14. Consider the equation $v'' + \left(\mu - z^2\right) v = 0$, where μ is a constant. Find the equation satisfied by u if $v = e^{-z^2/2}u$; this is called Hermite's equation. For what values of μ does Hermite's equation have polynomial solutions?

15. Chebyshev's equation is

$$\left(1 - z^2\right) u'' - zu' + \lambda u = 0.$$

The Chebyshev polynomials are solutions of this equation for certain values of λ. What are they?

16. In the equation of Problem 15 make the substiution $z = \cos\theta$ so that $u\,(z) = v\,(\theta)$ and find the equation satisfied by v. What are its solutions when λ is such that a Chebyshev polynomial exists?

4.5 Higher-order equations and systems

The nth-order equation (4.4) can be treated in exactly the same fashion as the second-order equation. The basic conclusion is that if R denotes the least of the radii of convergence of the coefficient functions $a_1, \ldots, a_n$, then any power-series solution of the equation has a radius of convergence not less than R. The reasoning in the following section establishes this.

4.5.1 The system of first-order equations

The equation (4.4) can be reformulated as a system of n first-order equations, exactly as in §2.3. The result is a system that may be written in vector-matrix form as

$$W' = A(z)\, W \tag{4.12}$$

where now W is a function of the complex variable z and has values in complex n-dimensional space, and $A(z)$ is an $n \times n$ matrix with coefficients that are analytic functions of z. This matrix has the special form of a companion matrix as in equation (2.36), but this special form will play no role in the considerations below; we may as well – and shall – assume we are given the system with no knowledge of the matrix A except that its entries are analytic functions of z.

A power-series solution of equation (4.12) may be sought in the form $\sum W_k z^k$, where now W_k is an n-component constant vector. Since the coefficient matrix A also has a power-series expansion $A(z) = \sum A_k z^k$, where A_k is a complex $n \times n$ matrix, the recursion relation for the coefficients W_k is found to be

$$(k+1)\, W_{k+1} = \sum_{j=0}^{k} A_j W_{k-j}. \tag{4.13}$$

This determines each W_{k+1} provided all its predecessors $W_0, \ldots, W_k$ are known. In particular, if W_0 is specified, then all subsequent coefficients are determined by this relation.

The convergence of this expansion can again be verified via Cauchy's method of majorants. We outline the steps without filling in all the details. Suppose each entry $a_{jk}(z) = \sum_l (a_{jk})_l\, z^l$ of the matrix A is majorized by a series $\sum_l (b_{jk})_l\, z^l$; form the matrix B having these entries. Then consider the equation

$$V' = BV, \tag{4.14}$$

and the corresponding recursion formula

$$(k+1)\,V_{k+1} = \sum_{j=0}^{k} B_j V_{k-j}. \tag{4.15}$$

Suppose the initial vector V_0 satisfies the condition $V_{0i} \geq |W_{0i}|$, $i = 1, \ldots, n$, i.e., the components of V_0 are at least equal in magnitude to those of W_0, term by term. It then follows by induction that for each k, $k = 0, 1, \ldots$, $V_{ki} \geq |W_{ki}|\,,\ i = 1, \ldots, n$. Thus the series $\sum V_{ki} z^i$ majorizes the series $\sum W_{ki} z^i$. For this reason the system (4.14) is said to majorize the system (4.12).

The strategy for establishing the convergence of the series $\sum W_k z^k$ is now, just as in the preceding section, to find a majorizing system and a solution of it whose initial data likewise dominate those prescribed for the system (4.12). If R is the least of the radii of convergence of the power-series for the n^2 functions $a_{ij}\,(z)$ of the matrix A, then for any number ρ with $0 < \rho < R$, each of those is majorized on a disk of radius $r < \rho$ by the power series $M \sum (z/\rho)^K = M\,(1 - z/\rho)^{-1}$ for some sufficiently large M. We may therefore take for B the matrix

$$M\left(1 - \frac{z}{\rho}\right)^{-1} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

We seek a solution of the majorizing system (4.14) in the form of a vector V all of whose components are the same, i.e.,

$$V\,(z) = v\,(z) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where v is complex-valued function of z, i.e., a scalar. For initial data we may choose

$$v\,(0) = \sum_{i=1}^{n} |W_{0i}|\,;$$

this ensures that the initial data for the majorant system appropriately dominate those for the system (4.12). It is easily checked that with this choice of initial data the majorizing system reduces to the solution of the one-dimensional system

$$v' = Mn\left(1 - \frac{z}{\rho}\right)^{-1} v,\ v(0) = \sum_{i=1}^{n} |W_{0i}|,$$

whose solution is

$$v(z) = \left(1 - \frac{z}{\rho}\right)^{-Mn\rho} \sum_{i=1}^{n} |W_{0i}|.$$

The power-series expansion of this function has radius of convergence ρ. It follows that the power series for W converges for $|z| < r < \rho$. But ρ and r can be arbitrarily close to R, so this power series converges for $|z| < R$. We have proved

Theorem 4.5.1 *In equation (4.12) suppose the entries of the matrix A have power series expansions about the origin with least radius of convergence R. Then, for arbitrary initial data W_0, the solution to that equation exists and has a power-series expansion with radius of convergence not less than R.*

4.5.2 The fundamental matrix solution

Theorem 4.5.1 asserts the existence of a power-series solution for the vector, or column matrix, $W(z)$ whose initial value W_0 is prescribed. The uniqueness of this solution is clear: if $W_0 = 0$ then $W_k = 0$ for all $k = 0, 1, 2, \ldots$. By linearity, if two solutions have the same initial data then their difference has zero initial data and they are the same.

There are n linearly independent choices of such an initial-value vector, according to the notions of linear independence in C^n. Let $W_0^{(1)}, W_0^{(2)}, \ldots, W_0^{(n)}$ be such a choice. Then the corresponding n solutions of the differential equation $W^{(1)}(z), W^{(2)}(z), \ldots, W^{(n)}(z)$ are necessarily linearly independent on the domain D where the solutions are defined (cf. 4.2.1). These n solutions, evaluated at any point z_* of the domain of their definition, are linearly independent as vectors in C^n. If they were not, there would be a linear combination of them that is zero. By uniqueness of solutions, this linear combination would remain zero in a neighborhood of z_* and therefore in the entire domain in which it is analytic.

Thus there is a fundamental matrix solution $\Phi(z)$ satisfying the matrix differential equation

$$\phi'(z) = A(z)\phi(z) \tag{4.16}$$

with initial value $\Phi(0)$ which is a nonsingular matrix. This matrix Φ is nonsingular throughout the domain where it remains analytic (at least a disk of radius R). As in the real case, the matrix $\Psi(z) = \Phi(z)K$, where K is constant matrix, satisfies equation (4.16) if Φ does. Therefore, if K is nonsingular, then Ψ is likewise a fundamental matrix solution, and any two fundamental matrix solutions are related in this way.

These remarks represent a direct extension of results of the real case to the complex case.

4.5.3 The inhomogeneous solution

The inhomogeneous equation can be treated like the homogeneous equation, via series solution as in §4.5.1. In place of the system (4.12) we now have the system

$$W' = A(z)\,W + \alpha(z) \tag{4.17}$$

where α is a given analytic function of z taking values in the complex n-dimensional space C^n; in other words, each component of α is an analytic function. We now denote by R the minimum of the radii of convergence not only for the entries a_{ij} of the matrix A but also for the components α_i of the column vector α. Each component of the latter is majorized by the power series for $M(1 - z/\rho)$ for any $\rho < R$, some sufficiently large M, and all z in the disk of radius r provided $r < \rho$. Denote by β the majorizing vector whose components are all identical and equal to $M(1 - z/\rho)$. Then the system

$$V' = BV + \beta, \tag{4.18}$$

with the same choice of B as in the preceding section, will lead to the first-order equation

$$v' = nM\left(1 - \frac{z}{\rho}\right)^{-1} v + M\left(1 - \frac{z}{\rho}\right)^{-1}.$$

This has the solution

$$v = \gamma\left(1 - \frac{z}{\rho}\right)^{-Mn\rho} - \frac{1}{n}.$$

The initial data must again dominate those for the problem (4.17). This can be achieved by choosing the constant γ large enough. The conclusion is the same: any power-series solution of the system (4.17) has radius of convergence at least equal to R.

Alternatively, the inhomogeneous equation may be treated as we did the real case, by the variation of parameters formula. Introducing a fundamental-matrix solution $\Phi(z)$, we can verify that the formula

$$W(z) = \Phi(z)C + \int_0^z \Phi(z)\Phi^{-1}(\zeta)R(\zeta)\,d\zeta, \tag{4.19}$$

where C is an arbitrary, complex, constant vector, represents the most general solution to the inhomogeneous problem.

4.6 The nonlinear, analytic equation

For nonlinear equations the method of majorants continues to be effective for proving the existence of solutions. The case of a single, first-order equation illustrates the technique.

Consider the nonlinear initial-value problem

$$w' = f(z, w),\ w(z_0) = w_0, \tag{4.20}$$

where w is sought as the complex-valued function of the complex variable z that is defined and satisfies the equation in a neighborhood of the point z_0, and takes on the value w_0 when $z = z_0$. The equation is assumed analytic in the sense that the function f is represented by the power series

$$f(z, w) = \sum_{k,l=0}^{\infty} a_{kl}\,(z - z_0)^k\,(w - w_0)^l \tag{4.21}$$

which converges at least for some values of $z \neq z_0$ and $w \neq w_0$. Without loss of generality we may simplify the problem slightly by choosing $z_0 = w_0 = 0$, since a shift of the variable achieves this. The problem then becomes

$$w' = f(z, w),\ w(0) = 0 \tag{4.22}$$

where

$$f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l, \tag{4.23}$$

and it is assumed that the indicated series converges for at least one value (z_*, w_*) where neither z_* nor w_* vanishes. From this latter requirement it follows that there exist real, positive numbers ρ, σ such that the series

$$\sum_{k,l}^{\infty} |a_{kl}| \rho^k \sigma^l$$

converges, and there is a constant M such that

$$|a_{kl}| \leq M\rho^{-k}\sigma^{-l},\ k,l = 0,1,2,\ldots \tag{4.24}$$

The reasoning is the same as that leading to the analogous conclusion in §4.1.

We seek a power-series solution

$$w(z) = \sum_{j=1}^{\infty} w_j z^j = w_1 z + w_2 z^2 + w_3 z^3 + \ldots.$$

The procedure for generating the coefficients in this series is the same as that for linear equations: substitute in equation (4.22), collect the terms of like powers of z, and set the resulting coefficient equal to zero. There are two important rules to verify in this procedure: the first is that it determines the coefficients $\{w\}_1^{\infty}$ recursively – i.e., w_k should depend only on $w_1, w_2, \ldots, w_{k-1}$; the second is that the expression for w_k should be a polynomial in $w_1, \ldots, w_{k-1}$ and the coefficients $\{a_{ij}\}$, with only *positive* coefficients. Let's examine the first few of these expressions:

$$\begin{aligned}
w_1 &= a_{00}, \\
w_2 &= (1/2)\left(a_{10} + a_{01}w_1\right), \\
w_3 &= (1/3)\left(a_{20} + a_{11}w_1 + a_{02}w_1^2 + a_{01}w_2\right), \\
w_4 &= (1/4)\left(a_{30} + a_{21}w_1 + a_{11}w_2 + a_{01}w_3 + a_{12}w_1^2 + 2a_{02}w_1w_2 + a_{03}w_1^3\right).
\end{aligned}$$

These first four coefficients conform to the rules, and it is not difficult to verify them generally[1].

Now define the function

$$F(z,W) = M \sum_{k=0,l=0} (z/\rho)^k (W/\sigma)^l; \tag{4.25}$$

the series is absolutely convergent if $|z| < \rho$ and $|w| < \sigma$, so the function F is defined in that domain of C^2. Moreover, it majorizes the corresponding series for $f(z,w)$. The initial-value problem

$$W' = F(z,W),\ W(0) = 0 \tag{4.26}$$

can be solved formally in exactly the same way that the formal series for $w(z)$ was obtained. Because of the second of the two rules noted above, it

[1]A fully detailed verification can be found in Hille, chapter 2.

follows that $|w_k| \leq W_k$ for $k = 1, 2, \ldots$. Therefore, if the series for W has radius of convergence R, then that for w has a radius of convergence of at least R. But the series for F is easily recognized as the product of the series for the functions $(1 - z/\rho)^{-1}$ with that of $(1 - W/\sigma)^{-1}$:

$$F(z, W) = M \left(1 - \frac{z}{\rho}\right)^{-1} \left(1 - \frac{W}{\sigma}\right)^{-1}.$$

With this choice of F, we can integrate equation (4.26) as follows:

$$W - (1/2\sigma)W^2 = -(M\rho)\ln(1 - z/\rho),$$

where we have used the initial condition to evaluate the constant term. A single-valued determination of the logarithm is obtained by introducing a branch cut along the real axis from $z = \rho$ to infinity. Solving this quadratic for W gives

$$W = \sigma \pm \left\{\sigma^2 + 2\sigma\rho\ln(1 - z/\rho)\right\}^{(1/2)}.$$

With a determination of the square-root function that makes it positive when its argument is real and positive, we need to choose the minus sign in order to satisfy the initial condition:

$$W = \sigma - \left\{\sigma^2 + 2M\sigma\rho\ln(1 - z/\rho)\right\}^{(1/2)}.$$

For $|z| < \rho$ this function is analytic as long as the argument of the square root does not lie on the negative, real axis. The smallest value of z for which this fails occurs with z real, say $z = r$ where

$$\sigma + 2M\rho\ln(1 - r/\rho) = 0,$$

or

$$r = \rho\left(1 - \exp(-\sigma/2M\rho)\right). \tag{4.27}$$

The majorizing series therefore converges with radius of convergence r. This proves

Theorem 4.6.1 *The initial-value problem defined by equations (4.22) and (4.23) has a solution that is analytic in a disk of radius $R \geq r$, where r is given by equation (4.27) in which the constants M, ρ, σ are those of equation (4.24).*

The estimate (4.27) for the radius of convergence is typically smaller than the actual radius of convergence of the series solution, but we can no longer expect the maximal result that was found in the linear case. In fact, the application of this estimate includes the linear case, and it is not hard to see that it provides an unnecessarily conservative estimate in this case.

This technique can be extended to prove the existence of analytic solutions to a system of n analytic equations: if

$$w'_k = \sum_{j=1}^{n} f_j(z, w_1, w_2, \ldots, w_n), \; w_i(z_0) = a_i$$

and the functions f_j can be expressed in convergent power series in their variables provided $|z| < \rho, |w_i| < \sigma_i, i = 1, 2, \ldots, n$. The bound M is the maximum of bounds M_i taken on by the series

$$\sum |a_{i,k_1,k_2,\ldots,k_n}| \rho^j \sigma_1^{k_1} \cdots \sigma_n^{k_n} = M_i.$$

We omit the details. Since the nth order equation

$$w^{(n)} = g(z, w, w', \ldots, w^{(n-1)})$$

can be reduced to an nth-order system, the analytic character of its solutions is a corollary of the corresponding result for the system.

PROBLEM SET 4.6.1

1. Generalize Theorem 4.3.1 to n functions analytic in a domain $D \subset C$.
2. Consider the equation $w' = w^2$ with initial data $w(z_0) = w_0$, and transform it to the form of equation (4.22). Compare the radius of convergence as estimated by eqution (4.27) with the exact radius of convergence of a power-series solution.
3. For any autonomous equation, i.e., one of the form $w' = f(w)$, provide an estimate for the radius of convergence of power-series solutions that is independent of the parameter ρ.
4. Consider the linear initial-value problem $w' = a(z)w + b(z), w(0) = 0$. Suppose that the functions a and b are analytic in a disk centered at the origin and of radius R. Compare the estimate of equation (4.27) with that which you know from the linear theory.

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