Chapter 10

Boundary-value problems

The initial-value problem is characterized by the imposition of auxiliary data at a single point: if the equation is of the $n$th order, the $n$ otherwise arbitrary constants in its solution are determined by prescribing the function and its first $n-1$ derivatives all at a single value of the independent variable. Boundary-value problems provide the auxiliary data by providing information, not at a single point, but at two points\footnote{Or possibly more than two, but the commonly arising case is that of two points.}, usually at the endpoints (or boundary) of an interval. They arise often in the solution of partial-differential equations.

**Example 10.0.2** Consider the temperature $u(x,t)$ in a wall of thickness $L$ whose interior, at $x = 0$, is maintained at a temperature $A$ and whose exterior, at $x = L$, is maintained at a temperature $B$. Initially, at $t = 0$, the distribution of temperature is $u = u_0(x)$. Its subsequent evolution is governed by the partial-differential equation (the *heat equation*)

$$
\rho(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( p(x) \frac{\partial u}{\partial x} \right). \tag{10.1}
$$

Here $\rho$ and $p$ are known functions depending on the construction of the wall, and both may be assumed positive. The problem of finding a solution of equation (10.1) together with the imposed initial data and boundary data is an example of an *initial-boundary-value problem*.

There is a simple, time-independent solution of this equation satisfying the boundary conditions:

$$
U(x) = A + (B - A) \frac{\int_0^x p(s)^{-1} \, ds}{\int_0^L p(s)^{-1} \, ds}. \tag{10.2}
$$
If we write \( u = U(x) + v \) in equation (10.1), we find that \( v \) satisfies exactly the same equation (10.1), but its boundary conditions now take the form \( v = 0 \) at \( x = 0 \) and at \( x = L \). Its initial value is \( v = v_0(x) = u_0(x) - U(x) \).

The method of separation of variables is an attempt to find a solution of equation (10.1) for \( v \), satisfying the boundary conditions, in the form \( v(x,t) = X(x)T(t) \) of a product of a function of \( x \) only with a function of \( t \) only. Substitution of this expression into equation (10.1) results in the equation \( \rho(x)X(x)T'(t) = T(t)(p(x)X'(x))' \). If one further assumes that the functions \( X \) and \( T \) are not zero except possibly at isolated points, then by dividing either side of this last equation by \( \rho XT \) one obtains

\[
\frac{T'(t)}{T(t)} = \frac{(p(x)X'(x))'}{\rho(x)X(x)}.
\]

Since the left-hand side depends on \( t \) only and the right-hand side on \( x \) only, they must be equal to the same constant, \( \lambda \) say.

This requires of these functions that

\[
\frac{d}{dx} \left( p(x) \frac{dX}{dx} \right) - \lambda \rho(x) X = 0 \quad (10.3)
\]

and

\[
\frac{dT}{dt} = \lambda T, \quad (10.4)
\]

where \( \lambda \) is some as yet undetermined constant. This will provide a solution \( v = X(x)T(t) \) of the heat equation if equation (10.3) has a solution satisfying the boundary data \( X(0) = 0, X(L) = 0 \). \( \square \)

This combination of the differential equation (10.3) with boundary data is called a boundary-value problem, to distinguish it from the initial-value problem. Unlike initial-value problems, boundary-value problems do not always have solutions, as we shall see below (cf. Example 10.1.1). Even if the boundary-value problem consisting of equation (10.3) does have a solution satisfying the boundary data, it will not in general provide a solution to the original initial-boundary-value problem, since the initial value \( T(0)X(x) \) will not in general agree with \( v_0(x) \). These deficiencies will be dealt with below.

Equation (10.3) of the preceding example is linear, homogenous and of the second order. We will mostly confine the discussion to this case. We begin with certain algebraic properties of linear operators.
10.1 Operators and their adjoints

Consider the general, linear, homogeneous operator $M$ of the second order defined by the formula

$$Mu = f(x) u'' + g(x) u' + h(x) u$$

(10.5)

on the interval $(a,b)$. The coefficients $f, g, h$ are assumed to be $C^2$ (i.e., continuous together with their first two derivatives). If we multiply $Mu$ by an arbitrary $C^2$ function $v$, integrate over the interval, and use integration by parts to move derivatives of $u$ to derivatives of $v$, we find

$$\int_a^b v Mu \, dx = \left\{ fv u' - (fv)' u + g uv \right\}_a^b + \int_a^b (M^* v) u \, dx,$$

where

$$M^* v = (fv)'' - (gv)' + hv = f v'' + (2f' - g) v' + (f'' - g' + h) v.$$  

(10.6)

The operator $M^*$ is said to be adjoint to the operator $M$. The relation between them can be summarized as follows:

$$\int_a^b (v Mu - u M^* v) \, dx = \left\{ f (vu' - uv') + (g - f') uv \right\}_a^b.$$  

(10.7)

The form on the right, which is bilinear in the values of $u, u', v, v'$ at the endpoints, is called the bilinear concomitant.

A case of special importance is that of the self-adjoint operator, $M^* = M$. For this to be true we must have

$$2f' - g = g \quad \text{and} \quad f'' - g' + h = h.$$  

This in turn holds precisely when $g = f'$; in this case the operator $M$ takes the form

$$Mu = \frac{d}{dx} \left( f(x) \frac{du}{dx} \right) + h(x) u.$$  

(10.8)

This is the form of the operator appearing on the left-hand side of equation (10.3), and is of very common occurrence in problems of mathematical physics, which is the origin of its special importance.

For the self-adjoint case, equation (10.7) can be written

$$\int_a^b (v Mu - u M v) \, dx = \left\{ f (vu' - uv') \right\}_a^b.$$  

(10.9)
The equation (10.3) of the preceding example has the form $Mu = \lambda \rho u$ where the operator of equation (10.8) is defined by $f = p$ and $h = 0$. The constant $\lambda$ that arose in the course of separating variables may appear arbitrary, unrelated to the problem to be solved, but this is not so. It cannot be assigned arbitrarily, or the problem of solving equation (10.3) together with the boundary conditions $X(0) = 0$ and $X(L) = 0$ will have only the trivial solution $X \equiv 0$.

**Example 10.1.1** In the Example 10.0.2, take $\rho(x) = p(x) = 1$. Then equation (10.1) takes the form $u_t = u_{xx}$ and equation (10.3) takes the form $X'' = \lambda X$. Solutions may be written $X = c \sin(\sqrt{-\lambda} x) + d \cos(\sqrt{-\lambda} x)$, where we have assumed that $\lambda \leq 0$; this will be verified subsequently. Imposition of the boundary condition $X(0) = 0$ then implies that $d = 0$, so imposition of the remaining condition at $x = L$ implies $c \sin(L\sqrt{-\lambda}) = 0$. If $c = 0$, the solution $X \equiv 0$. To have a nonzero solution we must have $L\sqrt{-\lambda} = \pi, 2\pi, \ldots$, i.e., $\lambda$ cannot be arbitrary but must take one or another of the discrete set of values $\lambda_n = -n^2\pi^2/L^2$.

There are therefore infinitely many separation solutions of the form

$$v_n(x,t) = e^{-n^2\pi^2 t/L^2} \sin(n\pi x/L).$$

Each satisfies the equation and the boundary conditions. Since the equation and the boundary conditions are linear and homogeneous, any linear combination with arbitrary coefficients $\{c_n\}$,

$$v(x,t) = \sum c_n v_n(x,t),$$

also satisfies the equation and the boundary conditions. If this sum can be made to satisfy the initial condition $v = v_0(x)$, a complete solution will be achieved. The latter condition is written

$$v_0(x) = \sum c_n \sin(n\pi x/L).$$

If the summation is allowed to run from 1 to $\infty$, this represents the expansion of the function $v_0$ in a Fourier sine series. Under appropriate conditions on the function $v_0$, this is indeed possible by choosing the coefficients $\{c_n\}$ to be the Fourier coefficients of $v_0$, although the convergence of the series for $v(x,t)$ requires further consideration. □
In the preceding example, the values of the parameter $\lambda$ for which non-trivial solutions exist are called eigenvalues, and the corresponding functions $\sin(\sqrt{-\lambda}x)$ are called eigenfunctions. Eigenvalues and their eigenfunctions play a central role in boundary-value problems. We therefore consider them in greater generality.

### 10.2 Eigenvalue problems

Consider the self-adjoint differential operator on the interval $[a, b]$, which we now write in the form

$$Mu \equiv -(p(x)u')' + q(x)u = \lambda \rho(x) u, \quad (10.11)$$

where the coefficients satisfy the following conditions:

$$p \in C^1 \text{ and } p > 0 \text{ on } [a,b]; \quad q, \rho \in C \text{ on } [a,b] \text{ and } \rho > 0 \text{ on } (a,b). \quad (10.12)$$

This form of the operator is general enough to apply to problems arising in diverse applications. That the coefficient $p$ is continuous and positive on the closed interval $[a,b]$ implies that it is bounded away from zero (i.e., that $p(x) \geq \delta$ for some $\delta > 0$ for each $x \in [a,b]$), and therefore the equations $Mu - \lambda \rho u = 0$ and $Mu - \lambda \rho u = r$ can be put into the standard form to which the theory of Chapter 2 applies. The minus sign has been introduced for later convenience: applied to Example 10.1.1, the new sign convention of equation (10.11) would make the eigenvalues turn out positive rather than negative. The integral relation (10.9) now takes the form

$$\int_a^b (vMu - uMv) \, dx = \left\{p(uv' - vu') \right\}_{a}^{b}. \quad (10.13)$$

The bilinear concomitant has been reduced to the product of the Wronskian of the two functions multiplied by the coefficient $p$.

The equation (10.11) must be supplied with boundary conditions. These we will take to be homogeneous boundary conditions generalizing those of Examples 10.0.2 and 10.1.1. The general structure of homogeneous boundary conditions is:

$$a_{11}u(a) + a_{12}u'(a) + b_{11}u(b) + b_{12}u'(b) = 0, \text{ and}$$
$$a_{21}u(a) + a_{22}u'(a) + b_{21}u(b) + b_{22}u'(b) = 0. \quad (10.14)$$

Here the numbers $\{a_{ij}, b_{ij}\}, \; i = 1, 2, \; j = 1, 2$ are prescribed: they determine the boundary data. Rarely is one called on to consider boundary data
of such generality. In the examples above, \(a_{11}\) and \(b_{21}\) are nonzero, and all the others vanish. But before passing on to more specific cases, we make a couple of observations that follow from the linear and homogeneous nature of both the equation and the boundary conditions. One is that \(u \equiv 0\) is always a solution of equation (10.11) together with the auxiliary data (10.14). Another is that if \(u\) is a solution so also is \(cu\) for any arbitrary constant \(c\) and, more generally, if \(u\) and \(v\) are solutions so also is any linear combination \(cu + dv\) with arbitrary constants \(c\) and \(d\).

Given an operator \(M\), we know how to construct the adjoint operator \(M^*\), as in equation (10.6) above. We may consider, along with the boundary-value problem

\[
Mu = \lambda \rho u, \quad a < x < b \text{ and conditions 10.14} \quad (10.15)
\]

the related boundary-value problem

\[
M^*v = \mu \rho v, \quad a < x < b \text{ and conditions 10.17,} \quad (10.16)
\]

where the boundary conditions for this related problem,

\[
a_{11}^* v(a) + a_{12}^* v'(a) + b_{11}^* v(b) + b_{12}^* v'(b) = 0, \text{ and}
\]

\[
a_{21}^* v(a) + a_{22}^* v'(a) + b_{21}^* v(b) + b_{22}^* v'(b) = 0, \quad (10.17)
\]

are similar in structure to equations (10.14) but with possibly different values for the coefficients \(\{a_{ij}^*\}\). Suppose now that we can choose these coefficients such that, whenever \(v\) satisfies the conditions (10.17) while \(u\) satisfies the conditions (10.14), the bilinear concomitant vanishes, i.e., by equation (10.7) above,

\[
\int_a^b vMu \, dx = \int_a^b uM^*v \, dx, \quad (10.18)
\]

for all \(C^2\) functions \(u, v\) satisfying conditions (10.14), (10.17), respectively. In this case, the eigenvalue problem (10.16) will be referred to as the adjoint problem to the eigenvalue problem (10.14).

That the adjoint boundary conditions can in fact be found such that the bilinear concomitant vanishes may be seen as follows. Let the four-component vector \(U\) be defined by the equations

\[
U = (u(a), u'(a), u(b), u'(b)) = (u_1, u_2, u_3, u_4),
\]
and similarly for $V$:

\[ V = (v(a), v'(a), v(b), v'(b)) = (v_1, v_2, v_3, v_4). \]

The pair of boundary conditions (10.14) may be written

\[ A_1 \cdot U = 0 \quad \text{and} \quad A_2 \cdot U = 0, \]

(10.19)

where $A_1$ and $A_2$ are appropriately chosen four-component vectors. These must be linearly independent if the two equations (10.14) are to represent two distinct boundary conditions, and we may therefore, without loss of generality, suppose they are orthogonal: \( A_1 \cdot A_2 = 0 \). We can extend these to an orthogonal basis \( A_1, A_2, A_3, A_4 \) for \( \mathbb{R}^4 \), and observe that

\[ U = \alpha A_3 + \beta A_4 \]

by virtue of the boundary conditions (10.19).

The bilinear concomitant, the right-hand side of equation (10.7), is bilinear in $U$ and $V$. If we call it $C$, we may write $C = V \cdot K U$ where $K$ is a four-by-four matrix whose entries can be read off equation (10.7). Therefore

\[ C = \alpha (V \cdot KA_3) + \beta (V \cdot KA_4). \]

The boundary conditions on $u$ are satisfied for any choices of $\alpha$ and $\beta$ so the vanishing of the bilinear concomitant $C$ requires that

\[ V \cdot KA_3 = 0 \quad \text{and} \quad V \cdot KA_4 = 0. \]

This implies there are at least two linearly independent choices of the vector $V$ that result in the vanishing of the bilinear concomitant, proving the claim.

If the matrix $K$ is nonsingular, there are precisely two, since the vectors $KA_3$ and $KA_4$ are then linearly independent: reading $K$ from equation (10.7) shows that it is indeed nonsingular, so there are precisely two.

We shall, beginning in the next section, restrict consideration to the case when one of the boundary conditions refers to the left-hand endpoint only, the other to the right-hand endpoint only, but we provide here simple examples of boundary conditions that each involve both endpoints.

**Example 10.2.1** Boundary-value problems are by no means confined to equations of the second order. A simple example is the first-order equation $u' = \lambda u$ on the interval $[0, 2\pi]$ with the single endpoint condition $u(0) = 2u(2\pi)$. If the operator $L$ is defined by $Lu = u'$ then the adjoint operator is
defined by \( L^*v = -v' \). It is easy to see that the adjoint boundary conditions is \( v(0) = (1/2)v(2\pi) \).

The solution is \( u = u_0 \exp(\lambda x) \) and the eigenvalue relation is \( \exp(2\pi \lambda) = 1 \). The permissible values of \( \lambda \) are therefore \( 0, \pm i, \pm 2i, \ldots \), all pure-imaginary. □

**Example 10.2.2** Consider the equation \( u'' + \lambda u = 0 \) on the interval \([0, 2\pi]\), with the boundary values \( u(0) - u(2\pi) = 0 \) and \( u'(0) - u'(2\pi) = 0 \). This problem is self-adjoint and the adjoint boundary conditions are the same as those above for \( u \). The following assertions can be verified explicitly:

- The solution is identically zero unless \( \lambda \) is equal to one or another of the eigenvalues \( 0, 1, 4, \ldots k^2, \ldots \).

- If so, the corresponding solutions (the eigenfunctions) are

\[ 1; \cos x, \sin x; \cos 2x, \sin 2x; \ldots; \cos kx, \sin kx; \ldots, \]

where to each eigenvalue \( k^2 \) except for \( k = 0 \) there correspond the pair of eigenfunctions \( \cos (kx), \sin (kx) \). □

By an eigenfunction we mean a nontrivial solution of a homogeneous boundary-value problem. This does not exist for general values of the parameter \( \lambda \): it requires that \( \lambda \) have one of a special set of values, i.e. an eigenvalue. The problem of finding these eigenvalues and eigenfunctions we refer to as an **eigenvalue problem**.

The definitions of the terms adjoint and self-adjoint that we have used are used in functional analysis in a manner that is related but not identical. There the notion of adjointness is introduced in connection with that of a linear functional on a linear space. A typical such functional may be written \( <v, u> \) where \( v \) labels the functional and \( u \) ranges over the linear space \( B \). If \( M : B_0 \to B \) is a linear operator on the space \( B_0 \), then \( <v, Mw> \) for \( w \in B_0 \) is again a linear functional on \( B_0 \), which is denoted by \( <M^*v, w> \). This defines the adjoint mapping \( M^* : B^* \to B_0^* \):

\[ <v, Mu> = <M^*v, u>. \]

The connection of this with our discussion above is as follows.

Suppose that the functions \( w(x) \) we wish to consider are in the space of \( C^2 \) functions on \([a, b]\) which satisfy the boundary conditions
This is a linear space; call it \( \mathcal{L}' \). Now, for a fixed function \( v(x) \) as yet unspecified, consider

\[
 f_v[u] = \int_a^b v(x)u(x)dx = (v, u)
\]

where \( u \in \mathcal{L} \), the space of continuous functions on \([a, b]\). This is easily seen to be a linear functional on \( \mathcal{L} \), for any fixed choice of \( v \). Then, noting that \( M : \mathcal{L} \to \mathcal{L} \) for the operator of equation (10.5), put \( u = Mw \). If we further restrict \( v \) to be in \( C^2 \) and to satisfy the adjoint boundary conditions (10.17), then, with our definitions of \( M \) and \( M^* \), we derive

\[
 (v, Mu) = (M^*v, u).
\]

Among the differences in the two approaches is one of generality: we do not consider all linear functionals on \( \mathcal{L} \), but only those defined by integration against a certain special class of functions \( v \).

### 10.3 The Sturm-Liouville problem

We shall emphasize the family of boundary-value problems in which the end-point conditions are separated: i.e., there is one condition at each endpoint, so the conditions (10.14) take the form

\[
 \alpha u(a) + \alpha' u'(a) = 0 \quad \text{and} \quad \beta u(b) + \beta' u'(b) = 0, \tag{10.20}
\]

where it is assumed that the coefficients \( \alpha \) and \( \alpha' \) are not both zero, and likewise for \( \beta, \beta' \). The assumption (10.20) excludes certain cases for which the endpoint conditions are not separated but are mixed, as in the more general conditions (10.14), or Examples 10.2.2 and 10.2.1 above.

The Sturm-Liouville problem is the eigenvalue problem consisting of the self-adjoint equation (10.11), under conditions on the coefficients given by equation (10.12), together with the separated endpoint conditions (10.20). It is the basic eigenvalue problem of ordinary differential equations, and takes its name from two of the mathematicians who developed its theory. We shall study the theory in greater depth in Chapter 11. In this section we establish some of its basic algebraic properties.

A simple but useful observation is the following.

**Lemma 10.3.1** Suppose two functions \( u \) and \( v \) both satisfy the endpoint condition (10.20) at \( x = a \). Then their Wronskian vanishes there.
Proof: The assumption means that
\[
\alpha u (a) + \alpha' u' (a) = 0, \quad \alpha v (a) + \alpha' v' (a) = 0.
\]
This is a system of two linear equations for \(\alpha\) and \(\alpha'\). The determinant of this system is the Wronskian of the functions evaluated at \(x = a\). If it were nonzero, we would arrive at the contradiction that \(\alpha\) and \(\alpha'\) both vanish. \(\square\)

Remarks:

- A similar conclusion holds for the other endpoint \(x = b\).

- If the functions \(u\) and \(v\) are allowed to be complex-valued functions of the real variable \(x\), this conclusion is unchanged.

We now easily find:

**Theorem 10.3.1** For the eigenvalue problem (10.11), (10.20) with separated endpoint conditions, there is at most one linearly independent eigenfunction corresponding to a single value of \(\lambda\).

Proof: Suppose there were two linearly independent solutions \(u\) and \(v\). They would both have to satisfy the endpoint condition at \(x = a\), so their Wronskian would have to vanish there. But they are both solutions of the equation (10.11) and if they were linearly independent their Wronskian could not vanish anywhere, hence they cannot be linearly independent. \(\square\)

Remarks:

- Example 10.2.2 above, for which the endpoint conditions are not separated, demonstrates that there may be two linearly independent eigenfunctions corresponding to a single eigenvalue in that case.

- We infer that for an eigenvalue problem with separated endpoint conditions, any eigenfunction is determined uniquely up to multiplication by an arbitrary constant.

- Example 10.2.1 shows that for some boundary-value problems, even though the coefficients in the equation and in the boundary conditions are real, the eigenvalues and eigenfunctions may be complex.

We shall show first that, for the Sturm-Liouville boundary-value problem, the eigenvalues are real. To establish this we must allow for the possibility that they be complex. We continue to restrict consideration to real
values of the independent variable $x$, and to real values of the coefficient functions $p, q, \rho$ and of the parameters $\alpha, \alpha'$ and $\beta, \beta'$ of the boundary conditions. If $\lambda$ is complex, the solution $u$ of equation (10.11) is necessarily also complex: $u(x) = u_1(x) + iu_2(x)$ where $u_1$ and $u_2$ are real. If we write $\overline{u} = u_1 - iu_2$ for the complex conjugate, we find $M\overline{u} = \overline{\lambda \rho u}$, whereas if $u$ satisfies the boundary conditions (10.20) so also does $\overline{u}$. We therefore have the two equations

$$Mu = \lambda \rho u, \quad M\overline{u} = \overline{\lambda \rho u}$$

where both $u$ and $\overline{u}$ satisfy the boundary conditions (10.20). Applying the integral condition (10.13) with $v = \overline{u}$ we find

$$\int_a^b (\overline{u}Mu - uM\overline{u}) \, dx = (\lambda - \overline{\lambda}) \int_a^b \rho |u|^2 \, dx = \{p(u\overline{u}' - \overline{u}u')\}_a^b.$$

The last term vanishes in view of Lemma 10.3.1 and we therefore find

$$(\lambda - \overline{\lambda}) \int_a^b \rho |u|^2 \, dx = 0.$$

If the integral vanishes, then $u \equiv 0$ on $[a, b]$: the integrand is non-negative and continuous, so if it should be nonzero at any point, there would be a neighborhood of that point throughout which it is positive, and this neighborhood would make a positive contribution to the integral, which would therefore be nonzero. Since $u$ is an eigenfunction, it cannot vanish identically and we conclude that the integral is nonzero, and therefore $\lambda = \overline{\lambda}$, i.e., $\lambda$ is real. We have proved

**Theorem 10.3.2** Any eigenvalue of the Sturm-Liouville problem is real.

In investigating eigenvalues, we can now restrict our attention to real values of $\lambda$. We can then also restrict attention to real-valued solutions $u$, since any complex-valued solution is obtained by multiplying a real-valued solution by a complex constant.

The integral $\int_a^b uv \, dx$ of the product of the functions $u$ and $v$ represents an analog of the inner product (or scalar product, or dot product) of vectors. It is an analogy that can be pushed quite far. It is therefore natural to refer to two functions for which this integral vanishes as orthogonal.

**Definition 10.3.1** The functions $u$ and $v$ are orthogonal on $(a, b)$ with respect to the weight function $\rho$ if

$$\int_a^b u(x) v(x) \rho(x) \, dx = 0. \quad (10.21)$$
Theorem 10.3.3  Eigenfunctions of the Sturm-Liouville problem belonging to different eigenvalues are orthogonal with respect to the weight function $\rho$.

Proof: Let $u$ and $v$ be eigenfunctions belonging to different eigenvalues $\lambda$ and $\mu$ respectively. Then

$$\int_{a}^{b} (vMu - uMv) \, dx = (\lambda - \mu) \int_{a}^{b} uv \rho \, dx = \{p (uv' - vu')\}_{a}^{b} = 0.$$ 

The conclusion now follows since $\lambda \neq \mu$. \hfill $\Box$

In Example 10.1.1 above, an important element was the ability to represent an essentially arbitrary function as an infinite sum of sine functions. More generally it is likewise important to be able to represent an essentially arbitrary function $f$ on $(a, b)$ in the form

$$f(x) = \sum_{n=0}^{\infty} c_{n} u_{n}(x)$$

where $\{u_{n}\}$ is a sequence of eigenfunctions of a Sturm-Liouville problem. If we assume such a development is possible and that we can interchange summation and integration, then the orthogonality of the eigenfunctions provides the coefficients in this sum:

$$c_{n} = \frac{\int_{a}^{b} f(x) u_{n}(x) \rho(x) \, dx}{\int_{a}^{b} u_{n}(x)^{2} \rho(x) \, dx}.$$ 

PROBLEM SET 10.3.1

1. Refer to Example 10.0.2. Verify

- that the function $U(x)$ given there is indeed a solution of equation (10.1),
- that if $u = U + v$ then $v$ indeed satisfies (10.1) if $u$ does and
- that $X(x)T(t)$ is indeed a solution of (10.1) if $X$ and $T$ are solutions of equations (10.3) and (10.4) respectively.

2. Work out $(M^*)^*$. 

3. For the operator $M$ given by the formula (10.5), find a function $I(x)$ such that $IM$ is self-adjoint. What do you need to assume about the coefficients of $M$?
4. In Example 10.1.1 there appears to be an implicit assumption that \( \lambda < 0 \). Assume instead that \( \lambda = \mu^2 \), where \( \mu \) is a real number. Can you achieve a nontrivial solution of the boundary-value problem of that example?

5. Prove the conclusions asserted in Example 10.2.2.

6. Consider the mixed-endpoint boundary-value problem

\[
-\frac{d}{dt} \left( p(t) \frac{du}{dt} \right) + q(t) u = \lambda u, \quad u(0) = u(T), \quad u'(0) = u'(T),
\]

where the coefficients \( p \) and \( q \) are defined and continuous for all real \( t \) and periodic with period \( T \): \( p(t + T) = p(t) \) and \( q(t + T) = q(t) \); in addition assume that \( p \) is positive and \( C^4 \). Show that any solution satisfying the given data is likewise periodic with period \( T \).

7. Suppose \( \lambda = \lambda_1 + i\lambda_2 \) and \( u(x) = u_1(x) + iu_2(x) \) represent a complex eigenvalue and eigenfunction, respectively, of the Sturm-Liouville problem \( M u = \lambda \rho u \) together with the boundary conditions (10.20). Write this problem as a pair of real differential equations for \( u_1 \) and \( u_2 \) and obtain Theorem 10.3.2 in this way.

8. Prove the statement in the text following Theorem 10.3.2 that if the eigenvalue \( \lambda \) is real then any complex eigenfunction is a complex multiple of a real eigenfunction.

9. The eigenfunctions of Example 10.1.1 are \( u_n(x) = \sin \left( \frac{n\pi x}{L} \right) \). Show by explicit integration that these are orthogonal on \((0, L)\) with weight function \( \rho(x) \equiv 1 \).

10. Consider the fourth-order operator

\[
M u \equiv p_0 u^{(4)} + p_1 u''' + p_2 u'' + p_3 u' + p_4 u.
\]

Assume that the coefficients \( p_i \), \( i = 0, \ldots, 4 \), are sufficiently differentiable and that \( p_0 \) does not vanish. Find the adjoint operator \( M^* \) by successive integrations by parts. Find the conditions on the coefficients for \( M \) to be self-adjoint, and re-express the self-adjoint operator in the form

\[
M u = \frac{d^2}{dx^2} \left( P \frac{d^2 u}{dx^2} \right) + \frac{d}{dx} \left( Q \frac{du}{dx} \right) + Ru,
\]
identifying the coefficients \( P, Q, R \) in terms of the original coefficients \( \{ p_i \} \). Show that if the boundary data are such that the corresponding bilinear concomitant vanishes, i.e., if

\[
\int_a^b (vMu - uMv) \, dx = 0,
\]

then the eigenvalues of any such self-adjoint operator are real.

11. For Example 10.2.1 the operator is \( M = d/dx \) and \( M^* = -M \) (verify this); \( M \) is said to be skew-adjoint. For the third-order linear operator

\[
Mu \equiv p_0 u''' + p_1 u'' + p_2 u' + p_3 u
\]

form the adjoint operator by successive integrations by parts, and show that the operator \( M \) cannot be self-adjoint, but can be skew-adjoint under appropriate conditions on the coefficients. Show further that if the boundary data are such that the corresponding bilinear concomitant vanishes, i.e., if

\[
\int_a^b (vMu + uMv) \, dx = 0,
\]

then the eigenvalues of any such skew adjoint operator are pure-imaginary.

12. Let \( \{ P_n (x) \}_{n=0}^\infty \) be a sequence of polynomials on \([a,b]\), with \( P_n \) of degree \( n \), and suppose they are orthogonal with respect to a positive weight function \( \rho \). Show that \( P_n \) has \( n \) zeros in \([a,b]\).2

### 10.4 Green’s Functions

Consider the general problem of solving the \( n \)th-order, inhomogeneous, differential equation

\[
Lu = r, \quad a < x < b \tag{10.23}
\]

where

\[
Lu \equiv p_0 (x) u^{(n)} + p_1 (x) u^{(n-1)} + \cdots + p_n (x) u
\]

and the coefficients \( r \) and \( \{ p_i \} \) are given, continuous functions on \([a,b]\) and \( p_0 \) does not vanish there. Now, however, we suppose the additional conditions

\[\text{\footnotesize \textsuperscript{2}This describes the Legendre polynomials, which indeed satisfy a differential equation of self-adjoint form, but the proof should be independent of the theory of differential equations}\]
on the solution $u$ and its derivatives to be given in the form of homogeneous boundary data like those of equation (10.14) rather than initial data. In particular, we suppose the boundary conditions to have the form

$$
\sum_{j=1}^{n} a_{ij} u^{(j-1)}(a) + b_{ij} u^{(j-1)}(b) = 0, \quad i = 1, 2, \ldots, n. \tag{10.24}
$$

We shall find that this boundary-value problem may fail to have a solution. However, if it does have a solution, it can be expressed in the form

$$
u(x) = \int_{a}^{b} G(x, \xi) r(\xi) \, d\xi, \tag{10.25}$$

where the function $G(x, \xi)$ appearing in the integrand depends on the operator $L$ and the boundary data, but not on the function $r$. The function $G$, called the Green’s function for this boundary-value problem, therefore plays a similar role to that played by the influence function for initial-value problems (cf. equation 2.26). We outline the principal result in §10.4.2.

It takes its simplest form in the (important) case of the Sturm-Liouville boundary-value problem, so we consider this in detail.

**10.4.1 The Sturm-Liouville Case**

Consider now the inhomogeneous equation

$$Mu - \lambda \rho u = r \tag{10.26}$$

with the same definition of the operator $M$ and the same separated boundary conditions as in §10.3; in particular, $M$ is given by equation (10.11). If $\lambda$ is an eigenvalue for the homogeneous problem and $v$ the corresponding eigenfunction, then

$$\int_{a}^{b} vr \, dx = \int_{a}^{b} v (Mu - \lambda \rho u) \, dx = \left\{ p (uv' - vu') \right\}_{a}^{b} + \int_{a}^{b} u (Mv - \lambda \rho v) \, dx = 0,$$

since the integral vanishes because of the assumptions on $v$ and the integrated part vanishes because $u$ and $v$ both satisfy the boundary conditions. We conclude that equation (10.26) can fail to have a solution: if $\lambda$ is an eigenvalue and $r$ is chosen so that $\int vr \, dx \neq 0$ (e.g., $r = v$), there can be no solution.

We then inquire whether there is always, i.e., for every continuous function $r$ on $[a, b]$, a solution of this boundary-value problem provided $\lambda$ is
not an eigenvalue of the homogeneous problem. Here the answer will be affirmative, and we can express the solution in the form

$$u(x) = \int_a^b G(x, \xi) \, r(\xi) \, d\xi. \quad (10.27)$$

The function $G(x, \xi)$ is called the Green’s function, and is the counterpart for boundary-value problems to the influence function and the variation-of-parameters formula obtained earlier (Chapter 2) in the discussion of linear initial-value problems. In fact, our derivation of the Green’s function, to which we now proceed, will be made to depend on that formula.

Let $u_1(x, \lambda)$ and $u_2(x, \lambda)$ be any basis of solutions of the homogenous differential equation

$$M u - \lambda p u = 0. \quad (10.28)$$

Then there is always a particular integral of equation (10.26) which may be written

$$U(x) = \int_a^x \left( \frac{u_2(\xi) \, u_1(x) - u_2(x) \, u_1(\xi)}{p(\xi) \, W(\xi)} \right) r(\xi) \, d\xi, \quad (10.29)$$

which is easily checked on writing the equation (10.26) in the standard form

$$u'' + \frac{p'}{p} u' + \frac{\lambda p - q}{p} = -\frac{r}{p}.$$

The solutions $u_1$ and $u_2$ depend on the parameter $\lambda$ but since $\lambda$ will not change in what follows, we suppress this dependence in equation (10.29) and in other relations below. Since the most general solution of equation (10.26) can be written in the form

$$u(x) = U(x) + c_1 u_1(x) + c_2 u_2(x) \quad (10.30)$$

for some choice of constants $c_1$ and $c_2$, this must also be true for the solution of the boundary-value problem (10.26) with the boundary conditions (10.20). It is just a matter of calculating $c_1$ and $c_2$.

The calculation is simplified somewhat if we choose the basis $u_1, u_2$ appropriately. In particular, we’ll choose $u_1$ to satisfy the left-hand boundary condition and $u_2$ to satisfy the right-hand boundary condition. This can always be done: for example, choose $u_1$ to solve the initial-value problem consisting of the differential equation (10.28) together with the initial data $u_1(a) = \alpha', \ u_1'(a) = -\alpha$, and similarly for $u_2$: it can be chosen to be a solution of equation (10.28) with initial data $u_2(b) = -\beta', \ u_2'(b) = \beta$. These
solutions are linearly independent: if they were linearly dependent we would have \( u_1(x) = Cu_2(x) \) where \( C \) is a constant, and then \( u_1 \) would satisfy both sets of boundary conditions and hence be an eigenfunction, contradicting the assumption that \( \lambda \) is not an eigenvalue.

It’s convenient to introduce a shorthand notation for the boundary conditions. For any \( C^1 \) function \( v \) on \([a,b]\) define
\[
A[v] = \alpha v(a) + \alpha' v'(a), \quad B[v] = \beta v(b) + \beta' v'(b).
\] (10.31)

In terms of these linear operators \( A \) and \( B \) the basis functions \( u_1, u_2 \) satisfy
\[
A[u_1] = 0, \quad B[u_2] = 0; \quad A[u_2] \neq 0, \quad B[u_1] \neq 0.
\]

Note also that our choice of particular integral \( U \) is such that \( U \) and \( U' \) both vanish at \( x = a \) and, as a result \( A[U] = 0 \). Now apply the boundary conditions \( A[u] = 0 \) and \( B[u] = 0 \) to the general solution (10.30) above. We find
\[
\]

This implies that \( c_2 = 0 \) and \( c_1 = -B[U]/B[u_1] \). But it is a straightforward calculation using the explicit form of \( U \) in equation (10.29) that
\[
B[U] = B[u_1] \int_a^b \frac{u_2(\xi)}{p(\xi) W(\xi)} r(\xi) \, d\xi,
\] (10.32)

This gives for the solution \( u \) the formula
\[
u(x) = -\int_a^x \frac{u_1(\xi) u_2(x)}{p(\xi) W(\xi)} r(\xi) \, d\xi - \int_x^b \frac{u_1(\xi)}{p(\xi) W(\xi)} r(\xi) \, d\xi,
\] (10.33)

identifying
\[
G(x,\xi) = \begin{cases} 
-u_2(x) u_1(\xi) / p(\xi) W(\xi) & \text{if } \xi < x \\
-u_2(\xi) u_1(x) / p(\xi) W(\xi) & \text{if } \xi > x
\end{cases}.
\] (10.34)

One can now verify that the formula (10.33) satisfies both the equation (10.26) and the boundary conditions (10.20). We now have the following

**Theorem 10.4.1** Suppose \( \lambda \) is not an eigenvalue of the Sturm-Liouville problem (10.11), (10.20). Then, for any continuous function \( r \) on \([a,b]\), there is a unique solution of equations (10.26), (10.20).
Proof: One solution \( u \) is provided by the formula (10.33). To see that it is unique, suppose there were a second \( v \). Then the difference \( u - v \) would satisfy equations (10.11), (10.20) of the Sturm-Liouville problem. Unless \( u - v \) is the zero solution, the latter has an eigenfunction, i.e., \( \lambda \) is an eigenvalue. This is a contradiction. \( \square \)

There is another characterization of the Green’s function for this problem that is often useful. Fix \( \xi \) in the interval \((a, b)\). Then \( G \) is the unique, continuous function satisfying the boundary conditions at each endpoint, satisfying the differential equation \( MG = 0 \) at each point of the interval except at \( x = \xi \), and satisfying also the jump-discontinuity condition

\[
\left[ \frac{\partial G}{\partial x} \right]_{\xi} = -1/p(\xi).
\]

Here the bracket notation means, for a function \( f(x) \),

\[
[f]_{\xi} = \lim_{x \to \xi^+} f(x) - \lim_{x \to \xi^-} f(x),
\]

i.e., the difference of the limiting values approached from the right and from the left. It is not difficult to check that the explicit expression (10.34) satisfies these conditions.

### 10.4.2 The General Case

The idea of the Green’s function can be generalized beyond the Sturm-Liouville boundary-value problem, and indeed beyond the theory of ordinary differential equations. In this brief section we merely outline the generalization to the boundary-value problem consisting of the differential equation (10.23) together with the linear, homogeneous boundary data (10.24). Note that the sign convention for the operator \( L \) of equation (10.23) differs from that of the preceding subsection.

We suppose that the coefficient \( p_0 \) does not vanish on the interval \([a, b]\). We suppose further that the homogeneous problem \( Lu = 0 \) together with the boundary data (10.24) has only the trivial solution \( u \equiv 0 \) on \([a, b]\). Under these conditions, one can define a function \( G(x, \xi) \) as follows:

**Definition 10.4.1** For fixed \( \xi \) in \((a, b)\), \( G \) satisfies the boundary data (10.24), it satisfies the differential equation \( Lu = 0 \) except at \( x = \xi \) and, at \( x = \xi \)

\[
\left[ \frac{\partial^k G}{\partial x^k} \right]_{\xi} = \begin{cases} 
0 & \text{if } k = 0, 1, \ldots, n - 2 \\
1/p_0(\xi) & \text{if } k = n - 1.
\end{cases}
\]
This function is the Green’s function for the boundary-value problem, i.e., the (unique) solution of the latter is given by the formula

\[ u(x) = \int_a^b G(x, \xi) r(\xi) \, d\xi. \]  \hspace{1cm} (10.35)

PROBLEM SET 10.4.1

1. Consider the differential equation \( M u - \lambda \rho u = 0 \) on \([a, b]\), but with the inhomogeneous boundary data

\[ \alpha u(a) + \alpha' u'(a) = A, \quad \beta u(b) + \beta' u'(b) = B. \]

Convert this to an inhomogeneous differential equation of the form (10.26) with homogeneous boundary data.

2. Verify equation (10.32).

3. Verify that the formula (10.33) provides a solution to equations (10.26) and (10.20).

4. Show that the denominator \( p(\xi) W(\xi) \) in equation (10.34) is constant (i.e., independent of \( \xi \)). Under what conditions does it vanish?

5. Find the Green’s function for the equation \( u'' + \lambda u = r(x) \) on the interval \([0, \pi]\) with boundary conditions \( u(0) = 0 \) and \( u(\pi) = 0 \). For what values of \( \lambda \) does it become singular?

6. Show that the Green’s function (10.34) is symmetric: \( G(x, \xi) = G(\xi, x) \).

7. Suppose that \( \lambda \) is an eigenvalue with eigenfunction \( u_1(x) \), and suppose the condition \( \int_a^b u_1r \, dx = 0 \) is satisfied. Show that if in equation (10.33) the function \( u_1 \) is this eigenfunction and \( u_2 \) is any linearly independent solution of the differential equation, the inhomogeneous boundary-value problem (10.26), (10.20) is satisfied.

8. Verify that the expression (10.35) solves the boundary-value problem (10.23), (10.24); show that the solution is unique.

9. Consider the linear operator \( Lu = -u' \) and the single boundary condition \( \alpha u(a) + \beta u(b) = 0 \). Under what conditions on \( \alpha \) and \( \beta \) is it true that the boundary-value problem \( Lu = 0 \) has only the trivial solution? Under this condition, work out the Green’s function for the solution of the inhomogeneous problem (10.23) with the same boundary condition.
10. The same as problem (9) but with $Lu = -u' + u$. 
Bibliography


