Chapter 7

Dynamical Systems

Mathematical models of scientific systems often lead to differential equations in which the independent variable is time. Such systems arise in astronomy, biology, chemistry, economics, engineering, physics and other disciplines. It is common to speak of a system of differential equations

$$\dot{x} = f(x) \tag{7.1}$$

as governing a *dynamical system* (or of *generating*, or of *being*, a dynamical system). We are more precise about this below (conditions 7.6).

In the present chapter we introduce some of the principal ideas of dynamical-systems theory, and illustrate them in the restricted but important context of planar systems, i.e., systems in R^2. A norm $\|\cdot\|$ will appear at various points below without explanation; we assume an appropriate norm has been chosen (see the discussion in §6.2). There is an emphasis on nonlinear systems,[1] and on:

1. the qualitative behavior of the system viewed on a long time interval. Put differently, the long-time dynamics, or the *asymptotic behavior* as $t \to +\infty$, is often a major issue.
2. behavior of solutions as the initial data are changed. In considering the system (7.1) of differential equations together with initial data $x(t_0) = a$, the solution is regarded not only as a function of t but also as a function of a: $x = \phi(t, a)$.

[1] However, as we shall see in Chapters 8 and 9, our previous emphasis on linear systems will prove to be of great importance.

3. families of differential equations. The members of the family are often distinguished from one another by the value of a parameter, μ (say). The right-hand side of equation (7.1) is then written as $f(x, \mu)$. The solutions will then depend also on μ and will sometimes be expressed as $x = \phi(t, a, \mu)$.

These features mimic those of problems arising in applications, which frequently depend on parameters, and which need to be considered for a variety of initial data.

> The one-dimensional equation $\dot{x} = \mu x$ with initial data $x(0) = a$ has the solution $x = a \exp(\mu t)$. Its behavior for $t \to +\infty$ depends dramatically on the sign of the parameter μ.
>
> The one-dimensional equation $\dot{x} = \mu x^2$ with initial data $x(0) = a$ has the solution $x = a/(1 - a\mu t)$. It's behavior depends dramatically on the parameters and initial data: if the product $a\mu < 0$, the solution decays to zero as $t \to \infty$; if $a\mu > 0$, it becomes unbounded in a finite time.

7.1 Autonomous Equations

The equations of the examples, or the system (7.1), are referred to as *autonomous* because the right-hand sides do not depend explicitly on the time. This terminology originates in mechanical examples in which an explicit time dependence reflects a mechanical forcing imposed upon the system by some external agent: when there is no such forcing the system is "self-governing," or autonomous. The more general, *non-autonomous*, system

$$\dot{x} = f(t, x) \tag{7.2}$$

of n equations may be converted to an autonomous system of $n+1$ equations by adjoining the equation $\dot{x}_{n+1} = 1$ and replacing t by x_{n+1} in the remaining n equations: this equivalent autonomous system is sometimes referred to as the *suspended* system. It may therefore appear that there is no difference between autonomous and non-autonomous systems, but this is not entirely true (see Problem 2 below).

In this chapter and in Chapters 8 and 9 we consider autonomous systems unless the opposite is explicitly stated.

A distinctive feature of autonomous systems is the arbitrariness of the origin of the time variable. If $x = \psi(t, t_0, p)$ represents the solution of the initial-value problem

$$\dot{x} = f(x), \; x(t_0) = p, \tag{7.3}$$

then

$$\psi(t, t_0, p) = \phi(t - t_0, p) \tag{7.4}$$

where $\phi(t, p) = \psi(t, 0, p)$, i.e., ϕ is the solution of the initial-value problem when $t_0 = 0$. This is a straightforward consequence of the uniqueness theorem for the initial-value problem as discussed in Chapter 6: either side of the preceding equation is seen to satisfy the initial-value problem, and they must therefore be the same (see problem 2 of Problem Set 6.4.1).

The solution map $\phi(t, p)$ of (7.3), sometimes called the *flow*, can be thought of as a family of mappings, parametrized by the time t, defined on the domain D_0 of the initial-value vectors p in R^n, and taking values in another domain D_t of R^n, namely, the image of the domain D_0 under the mapping $\phi(t, \cdot)$. As long as the initial-value problem continues to be satisfied at points of the latter domain, the identity

$$\phi(t + \alpha, p) = \phi(t, \phi(\alpha, p)), \tag{7.5}$$

holds, again as a direct consequence of the uniqueness theorem.

A formal definition of *dynamical system* that is sometimes used in abstract studies is that of a family of maps $\phi(t, \cdot) : \Omega \to \Omega$, defined on a domain Ω parametrized by a variable t on an interval (a, b) and satisfying, for each $p \in \Omega$ the following three conditions:

$$(i)\, \phi(0, p) = p \quad (ii)\, \phi \in C[(a, b) \times \Omega] \quad (iii)\, \phi(t + \alpha, p) = \phi(t, \phi(\alpha, p)). \tag{7.6}$$

If in item (ii) the C is replaced by C^k with $k \geq 1$, the dynamical system is said to be differentiable. These studies can be viewed as abstractions of flows of autonomous differential equations. When the mapping ϕ is generated by an initial-value problem (7.3, it is a *homeomorphism*, i.e., a continuous map with a continuous inverse. If the vector field f is C^k, it is a diffeomorphism[2].

There are also *Discrete* dynamical systems for which t takes on discrete values, but we shall consider continuous dynamical systems generated by autonomous systems like equation (7.1) – for which t takes on values in an interval of the real axis – unless otherwise specified.

[2]A diffeomorphism is differentiable map that possesses a differentiable inverse.

Suppose for each $p \in R_0$, a subdomain of Ω, solutions exist on a common interval (a, b), then the family of maps $\phi(t, \cdot)$ generates a homeomorphism from R_0 to its image R_t under the map for any $t \in (a, b)$.

The *orbit* $\gamma(p)$ through a point p is defined as the set

$$\gamma(p) = \{x \in R^n : x = \phi(t, p) \text{ for some } t \in (a, b)\}. \tag{7.7}$$

Here (a, b) will always be the maximal interval unless otherwise stated; in general it depends on the point p. The values $a = -\infty$ and $b = \infty$ are allowed for solutions defined on infinite intervals. Note that two orbits either coincide or do not intersect. For if $q \in \gamma(p)$ then $q = \phi(t', p)$ for some $t' \in (a, b)$. Then $\phi(t, q) = \phi(t, \phi(t', p)) = \phi(t + t', p)$. Thus $\gamma(q) = \gamma(p)$. This property of autonomous systems – that they either coincide or do not intersect – is not shared by nonautonomous systems, as the following example shows.

Example 7.1.1 The nonautonomous system

$$\dot{x} = -x + \epsilon(\cos t - \sin t), \quad \dot{y} = y + \epsilon(\cos t + \sin t)$$

has solutions $x = ae^{-t} + \epsilon \cos t$, $y = be^t + \epsilon \sin t$. If we choose $a = b = 0$, the orbit is a circle of radius ϵ. However, if we take instead initial data $(x(0), y(0)) = (0, \epsilon/2)$ lying inside the circle, then $x = \epsilon(\cos t - e^{-t})$ and $y = \epsilon((1/2)e^t + \sin t)$, which leaves the disk of radius ϵ and hence intersects the circle at some $t > 0$. □

Of course, if distinct orbits of nonautonomous systems cross, they must cross at different times, to be consistent with the general uniqueness theorem.

7.2 Constant and Periodic Solutions

The simplest kinds of solutions are *equilibrium points*. These are points where $f(p) = 0$. The solution map $\phi(t, p) = p$ for all t, i.e. p is a fixed point of the map for each t. The orbit $\gamma(p)$ is the set whose sole member is p.

Theorem 7.2.1 *If p is an equilibrium point in Ω and $\phi(t, q) \to p$ as $t \to t_0$, then either $t_0 = \pm\infty$ or $q = p$.*

Proof: If t_0 is finite then, since $p \in \Omega$, the solution ϕ can be continued to t_0, and $\phi(t_0, q) = p$ by continuity. This means that the orbit through q is the orbit through p, and since p is an equilibrium point, coincides with p. □

The next simplest kind of solution is periodic: $\phi(t + T, p) = \phi(t, p)$ for some T and all $t \in R$. We may assume that $T > 0$.

Theorem 7.2.2 *A nonconstant periodic solution has a least period T.*

Proof: Suppose not; then there is a sequence $\{T_n\}$ of periods, with $T_n \to 0$ as $n \to \infty$. For arbitrary t we have

$$\dot{\phi}(t, p) = \lim_{n \to \infty} \frac{\phi(t + T_n, p) - \phi(t, p)}{T_n} = 0.$$

But this would imply that $\phi(t, p)$ is constant, which is a contradiction. □

A nonconstant periodic solution has an orbit which is a simple closed curve: the solution map $\phi(t, p)$ for fixed p maps the interval $[0, T]$ to its image in a one-one fashion if 0 and T are identified. Here T is the least period. A criterion for a solution to be periodic is given by the following theorem.

Theorem 7.2.3 *Let ϕ be a solution of* (7.1) *that intersects itself i.e. $\phi(t_1) = \phi(t_2)$ with $t_1 < t_2$. Then ϕ is periodic with period $t_2 - t_1$.*

Proof: With $T = t_2 - t_1$ we have $\phi(t + T, p) = \phi(t + t_2 - t_1, p) = \phi(t - t_1, \phi(t_2, p)) = \phi(t - t_1, \phi(t_1, p)) = \phi(t, \phi(0, p)) = \phi(t, p)$. □
In this theorem, $T = t_2 - t_1$ need not be the least period.

7.3 Invariant Sets

A subset S of the domain Ω is an *invariant set* for the system (7.1) if the orbit through a point of S remains in S for all $t \in R$. If the orbit remains in S for $t > 0$, then S will be said to be *positively invariant*. Related definitions of sets that are *negatively invariant*, or *locally invariant*, can easily be given.

If p is an equilibrium point of the system (7.1) then the set consisting of p alone is an invariant set for that system. Likewise, if $\gamma(p)$ is a periodic orbit, it too is an invariant set. Invariant sets that are of particular importance in the remainder of the present chapter are the so-called $\omega-$ and $\alpha-$ limit sets, to which we now turn.

Suppose the solution $\phi(t,p)$ exists for $0 \le t < \infty$. The *positive semi-orbit* $\gamma(p)$ is defined by equation (7.7) above when (a,b) is replaced by $[0,\infty)$. The $\omega-$limit set for such an orbit $\gamma(p)$ is defined as follows

$$\omega[\gamma(p)] = \{x \in R^n : \exists \, \{t_k\}_{k=0}^{\infty} \text{ with } t_k \to \infty, \text{ such that } \phi(t_k,p) \to x \, \text{as} \, k \to \infty\}.$$

It is clear that the solution must exist at least on $[0,\infty)$ for this definition to make sense. For solutions that exist on $-\infty < t \le 0$ the negative semi-orbit and the α-limit set are defined similarly.

Example 7.3.1 Suppose $\gamma(p)$ is a periodic orbit and let q be any point of it. Then $q = \phi(\tau,p)$ for some $\tau \in [0,T)$, where T is the least period, and $\phi(t_k,p) = q$ if $t_k = \tau + kT$. Hence every point of the orbit is a point of the ω-limit set, i.e., the orbit equals the ω-limit set in this case: $\omega[\gamma(p)] = \gamma(p)$. It's also true that $\alpha[\gamma(p)] = \gamma(p)$.

The α- and ω-limit sets have a number of basic properties that will be needed in the next section.

Lemma 7.3.1 *The $\omega-$limit set is closed.*

Proof: Pick a sequence $q_n \in \omega$ with $q_n \to q$. By the definition of the ω-limit set, we know there exists $\{t_{n,k}\}$, $t_{n,k} \to \infty$ with $\phi(t_{n,k},p) \to q_n$ as $k \to \infty$. For each n, pick $K(n)$ so that $\|\phi(t_{n,k},p) - q_n\| < 1/n \; \forall \, k > K(n)$. Then, given $\epsilon > 0$, pick N so that $\|q - q_n\| < \epsilon/2$ if $n > N$. This implies that

$$\begin{aligned} \|\phi(t_{n,K(n)},p) - q\| &\le \|\phi(t_{n,K(n)},p) - q_n\| + \|q - q_n\| \\ &< 1/n + \epsilon/2 \\ &< \epsilon \quad \text{if } n > \max\{N, 2/\epsilon\}. \end{aligned}$$

Thus q is in ω. $\square$

Lemma 7.3.2 *Suppose $\omega(\gamma(p))$ is bounded and lies in Ω. Then ω is an invariant set.*

Proof: Let q lie in $\omega(\gamma(p))$. We need to show that $\phi(t,q) \in \omega[\gamma(p)]$ for all $t \in R$. We know there exists a sequence $\{t_k\}$ such that $\phi(t_k,p) \to q$ as $t_k \to \infty$. Then certainly $\phi(t+t_k,p) = \phi(t,\phi(t_k,p)) \to \phi(t,q)$ as $t_k \to \infty$. Therefore $\phi(t,q) \in \omega[\gamma(p)]$ on its maximal interval of existence. Since $\omega[\gamma(p)]$ is closed (by the preceding lemma) and bounded (by assumption) we infer that the latter is $(-\infty,\infty)$, as follows. If the left-hand endpoint of the maximal interval (a, say) is finite, then as $t \to a+, \phi(t,q)$ would have to approach a boundary point of Ω, by Theorem 6.1.5. This is not possible because this limit is confined to $\omega(\gamma(p))$, which cannot intersect the boundary of the domain Ω. $\square$

The following theorem summarizes the properties of the ω-limit set. We use the notations

$$d[x,y] = \|x-y\| \tag{7.8}$$

to represent the distance between the points x and y in terms of the norm $\|\cdot\|$. Similarly, if A is a set and x a point,

$$d(x,A) = \inf_{y \in A} d(x,y)$$

with a similar definition for the distance $d(A,B)$ between two sets A and B.

Theorem 7.3.1 *Let $\gamma(p)$ be a bounded, positive semi-orbit whose closure lies in Ω. Then $\omega[\gamma(p)]$ is non-empty, compact, invariant, connected and*

$$d[\phi(t,p),\omega(\gamma(p))] \to 0 \quad \textit{as} \quad t \to \infty. \tag{7.9}$$

Proof: The sequence $\{\phi(k,p)\}$ is bounded, hence contains a convergent subsequence. This shows that $\omega[\gamma(p)]$ is non-empty.

Since $\gamma(p)$ is bounded, so is the set of its limit points, so $\omega(\gamma(p))$ is bounded. We showed earlier that it is closed. This implies the compactness.

Since the closure of $\gamma(p)$ lies in Ω, so too does $\omega[\gamma(p)]$ and, by the previous lemma, it is invariant. It remains to prove the connectedness and the condition (7.9). We consider the latter first.

Suppose the condition (7.9) fails. Then there exists $\epsilon > 0$ and $\{t_k\} \to \infty$ such that $d[\phi(t_k,p),\omega(\gamma(p))] > \epsilon$. Now the boundedness of the orbit guarantees the existence of a convergent subsequence of $\{\phi(t_k,p)\}$; continue to denote this subsequence $\{\phi(t_k,p)\}$ and the corresponding subsequence of t-values $\{t_k\}$. Let $q = \lim_{k\to\infty} \phi(t_k,p)$. Then $q \in \omega(\gamma(p))$, but $d[q,\omega(\gamma(p))] \geq \epsilon$. This is obviously a contradiction.

It remains to show that ω is connected. We use the two following easily verified assertions. One of these is that the distance $d(x, A)$ from a point x to a set A is a continuous function of x. A second refers to the description of a set which is not connected. The usual description is that ω is not connected if there exist open sets A, B with the following properties: (i)$A \cap B = \phi$, (ii) $A \cup B \supset \omega$ and (iii) $A \cap \omega \neq \phi$ and $B \cap \omega \neq \phi$. The assertion is that if ω is closed and these properties hold for open sets A and B, they hold also for closed sets A' and B'. This is easily seen by checking the properties for the closed sets $A' = A \cap \omega$ and $B' = B \cap \omega$.

We now consider the connectedness of ω. Suppose it is not connected; then there exist closed sets A, B with the properties described above. Since A, B are disjoint closed sets, they are some finite distance apart. Denote $d[A, B] \equiv \delta > 0$. Since there is some point of ω in $A, \exists t_1$ such that $d[\phi(t_1), A] < \delta/4$. By the same reasoning, there is some time $t_2 > t_1$ such that $d[\phi(t_2), B] < \delta/4$. Again, there is some $t_3 > t_2$ such that $d[\phi(t_3), A] < \delta/4$. Continuing in this manner we obtain a sequence $\{t_k\}$ such that alternatively for odd and even indices, the orbital point $\phi(t_k)$ lies within a distance $\delta/4$ of A or of B; we can clearly arrange this sequence so that $t_k \to \infty$. Note that

$$\delta = d[A, B] \leq d[\phi(t_1), A] + d[\phi(t_1), B] \Rightarrow d[\phi(t_1), B] \geq 3\delta/4.$$

Since the distance function is continuous, we can find some τ_1 in $[t_2, t_1]$ such that $d[\phi(\tau_1), B] = \delta/2$. It follows, as above, that $d[\phi(\tau_1), A] \geq \delta/2$. We repeat this process for $\tau_k \in [t_{2k+1}, t_{2k}] \, \forall \, k \geq 0$. This process generates another sequence $\{\tau_k\}$ with $\tau_k \to \infty$ and

$$d[\phi(\tau_k), A] \geq \delta/2, \quad d[\phi(\tau_k), B] = \delta/2.$$

Since $\{\phi(\tau_k)\}$ is bounded, there exists a convergent subsequence; continue to denote this subsequence with the same notation. Let $q = \lim_{k\to\infty} \phi(\tau_k)$. Then $q \in \omega$, but q is bounded away from both A and B. This contradiction completes the proof. □

The ω-limit set and α-limit set are two of a number of invariant sets that are encountered in characterizing the qualitative behavior of orbits. Here is another.

Definition 7.3.1 *A point p is* nonwandering *for the system (7.1) if, given any neighborhood U of p and any time $T > 0, \phi(t, x) \in U$ for some $x \in U$ and some time $t \geq T$. The set of such points in Ω is called the* nonwandering set.

Example 7.3.2 Let Ω be bounded, and suppose the system (7.1) generates a mapping preserving Lebesgue measure[3]. For any $p \in \Omega$ choose a neighborhood U and a time T as in the definition. Consider the family of mappings $\{\phi(T, U), \phi(2T, U), \ldots, \}$. Each of these has the same, positive measure. If they were all disjoint, the measure of Ω would not be finite, but we have assumed that it is in assuming that Ω is bounded. Therefore two of these have a non-empty intersection, say $\phi(mT, U)$ and $\phi(nT, U)$, where $m < n$. This implies a non-empty intersection of U and $\phi\left((n-m)\,T, U\right)$. Therefore for measure-preserving mappings, every point is nonwandering.

Important invariant sets may be specific to a problem, as in the next example.

Example 7.3.3 In ecological modeling the *Lotka-Volterra* system is often introduced to approximate the relation between a predator and its prey in some fixed locale. Here the variable x is proportional to the number of prey (rabbits, say) in this locale whereas y is proportional to the number of predators (foxes, say). The equations are

$$\dot{x} = x(a - by), \quad \dot{y} = -y(c - dx), \tag{7.10}$$

where a, b, c, d are positive constants. Here it is important for the ecological interpretation that x, y be positive. In other words, if initially this is so, then this property is preserved by the equations. This is in fact the case (see Problem 7 below). □

PROBLEM SET 7.3.1

1. Prove the identity (7.5)
2. Show that the suspended system of dimension $n+1$ obtained from the time-dependent system (7.2) of dimension n cannot have an equilibrium point.
3. Find the periodic solutions of the system

   $$\dot{x}_1 = -x_2 + x_1 f(r), \; \dot{x}_2 = x_1 + x_2 f(r)$$

   where $r^2 = x_1^2 + x_2^2$ and $f(r) = -r(1-r^2)(4-r^2)$.

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[3]A Hamiltonian system will do this.

4. Consider the *nonautonomous*, periodic system

$$\dot{x} = f(x,t), \quad f(x,t+T) = f(x,t). \tag{7.11}$$

Let $x(t)$ be a solution such that, at some time t_1, $x(t_1) = x(t_1 + T)$. Show that this solution is periodic with period T.

5. Consider the planar autonomous system

$$\frac{dx}{dt} = f(x), \; x \in \Omega \in R^2,$$

and suppose

$$\text{div} f = \partial f_1/\partial x_1 + \partial f_2/\partial x_2$$

has one sign in Ω. Show that this system can have no periodic orbits other than equilibrium points.

6. Consider the gradient system

$$\frac{dx}{dt} = \nabla\phi, \; x \in \Omega \in R^n,$$

where $\phi(x)$ is a smooth, single-valued function. Draw the same conclusion as in the preceding problem.

7. Consider the system

$$\dot{x} = xf(x,y), \; \dot{y} = yg(x,y)$$

where f, g are arbitrary, smooth functions defined in R^2. Show that the lines $x = 0$ and $y = 0$ are invariant curves for this system. Infer that each of the four quadrants of the xy-plane is an invariant region for this system.

8. Show that the nonwandering set is closed and positively invariant.

7.4 Poincarè-Bendixson Theory

The behavior of solutions of the system (7.1) is severely circumscribed when $n = 2$ where it takes the form

$$\dot{x}_1 = f_1(x_1,x_2), \quad \dot{x}_2 = f_2(x_1,x_2). \tag{7.12}$$

Figure 7.1: Figure (a) indicates an orbit γ intersecting a transversal S at the point p and returning to S at q, forming a Bendixson pocket. Here γ spirals outward along S. Figure (b) is the same except that the orbit spirals inward along S.

This is due to the limited possibilities for orbits in the plane. A closed orbit, representing a periodic solution, is a Jordan curve, i.e. the topological image of a circle. The separation property of such a curve is based on the Jordan curve theorem, which states that if J is a Jordan curve in R^2, then the complement of J is the union of two disjoint open sets G_i and G_e, each of which has J as a boundary. G_i is bounded and called the interior of J, whereas G_e is unbounded and called the exterior of J. In the present section we consider the case $n = 2$, taking for granted the Jordan curve theorem. The resulting theory is usually referred to as Poincarè-Bendixson theory. One statement of its principal result is

Theorem 7.4.1 *Suppose the system (7.12) has dimension two, and is defined and C^1 on a compact set K containing no equilibrium points of f. If this system has a positive semi-orbit γ remaining in K for all $t > 0$, then it has a periodic orbit in K.*

The proof requires some development and is completed below following a series of lemmas.

We need the notion of a *transversal* to orbits of the system (7.12):

Definition 7.4.1 *A finite closed segment S of a straight line in R^2 is called a transversal with respect to f if $f(x) \neq 0$ for every point $x \in S$ and if the direction determined by f at every point of S is different from that of S.*

Figure 7.1 provides an illustration.

We infer that every orbit that meets a transversal must cross it, and that all such orbits must cross it in the same direction. Otherwise f would have to be tangent to S somewhere along it, i.e., determine a direction the same as that of S. It is clear that a transversal can be constructed at *any* regular point of f.

We can construct a *flow box* B on a transversal S consisting of orbits originating on S and continued throughout a small time interval $[-\sigma, \sigma]$: σ is small enough that the time interval required for a point of S to return to S along an orbit of (7.12) exceeds σ. If $q \in B$, then under the flow ϕ of (7.12) there is a unique point $q^* = \phi(t^*, q)$ on S at a unique time $t^* \in [-\sigma, \sigma]$. We'll return to a more formal justification of this in Chapter 9; here we rely on our intuition in the plane and view it as obvious.

Lemma 7.4.1 *If γ is not periodic and meets the transversal S_0 to y_0 at distinct points $y_k = \phi(t_k, x)$ at times $t_1 < t_2 < \ldots$, then the order of the points y_k on S_0 is the same as the order of the times t_k, i.e. y_k lies between y_{k-1} and y_{k+1}. If γ is periodic, it can meet S at most at one point.*

Proof: Suppose first that γ is not periodic. Let Σ be the simple closed curve consisting of γ between y_0 and y_1 and the segment of S_0 that joins y_0 and y_1 (see Figure 7.1 (b) to fix ideas). The orbit enters the Bendixson pocket at y_1 in the same direction as at y_0. The orbit can never leave the Bendixson pocket: leaving by crossing γ would imply periodicity, and leaving by crossing S_0 is not possible because the vector field is in the wrong direction along S_0. Hence the next crossing must occur beyond y_1, i.e. y_1 lies between y_0 and y_2. Repeating this reasoning shows that y_{k+1} lies between y_k and y_{k+2}. Next suppose that γ is periodic, with least period T. If it crosses S at distinct points y_0 and y_1 at times t_0 and $t_1 > t_0$, we may assume that $t_1 < t_0 + T$. Again forming the Bendixson pocket, we find that the orbit cannot return to y_0 by crossing S so must do so by crossing γ before reaching y_1 and therefore at a time earlier than t_1. This would imply periodicity with a period less than T, a contradiction. $\square$

Remark: If the orbit is traversed as in Figure 7.1(a), then the orbit *leaves* the Bendixson pocket instead of entering it, and cannot return. The conclusion is the same as that drawn above.

Lemma 7.4.2 *Let p be a regular point of $\omega(\gamma)$ lying in Ω. It is possible to construct a transversal S to γ at p . If γ is not periodic, then it must* *intersect S at infinitely many distinct points. If these intersections with S are denoted by p_i, then they all lie on the same side of p and the sequence $\{p_i\}$ converges monotonically to p.*

Proof: One can construct a transversal to $\omega(\gamma)$ at p. Construct a flow box B on S including p. Since p is a limit point of γ the latter must enter B and therefore intersect S, showing that S is a transversal also to γ. These intersections must be distinct since otherwise γ would be periodic. Since they are all in the same order according to Lemma 7.4.1 they must lie on the same side of p. Since the latter is in ω, one can, by constructing successively smaller flow boxes containing p, infer that they must tend toward p. □

Lemma 7.4.3 *If γ and $\omega(\gamma)$ have a regular point in common, then γ is a periodic orbit.*

Proof: Let $p_1 \in \gamma \cap \omega(\gamma)$. Construct a transversal S and a flow box at p_1. Since $p_1 \in \omega(\gamma)$, the orbit must return to the flow box and hence to the transveral at some later time t_2. Let $p_2 = \phi(t_2, p_1)$ and assume that $p_2 \neq p_1$ and therefore, by Lemma 7.4.2, that γ is not periodic. Since p_1 is a limit point of γ, the orbit intersects S infinitely often. Let $p_3 = \phi(t_3, y_1)$ be the next intersection of the orbit with S. By Lemma 7.4.1 p_3 must occur "further along" S from p_1 than p_2. Consruct the Bendixson pocket based on p_2 and p_3. Now the orbit γ cannot approach p_1 for $t > t_3$. However, this is clearly a contradiction since $p_1 \in \omega(\gamma)$. □

Lemma 7.4.4 *A transversal to γ cannot meet $\omega(\gamma)$ in more than one point.*

Proof: Suppose S is a transversal to γ, and let $p_1 \neq p_2$ be two points of $\omega(\gamma)$ lying in S. Then the orbit γ also meets S at more than one point, and is therefore not periodic (Lemma 7.4.1). Then γ meets S at infinitely many points $\{y_k\}$ at times $t_1 < t_2 < \cdots$. However, since there are two limit points of γ on S, it is impossible for the points $\{y_k\}$ to be in the corresponding order on S, contradicting Lemma 7.4.1.

Lemma 7.4.5 *Suppose γ is bounded and $\omega(\gamma)$ lies in Ω and contains a non-constant periodic orbit γ_0. Then $\omega(\gamma) = \gamma_0$.*

Proof: Let $d = \omega(\gamma)\backslash\gamma_0$ and suppose this set is non-empty. Since ω is connected, there exists a limit point of d in γ_0 (see Remark below): $q_k \in$

d, $q_k \to q_0$, $q_0 \in \gamma_0$. Now, $q_0 \in \omega$, and is a regular point since otherwise the periodic orbit γ_0 would be constant. By Lemma 7.4.2, one can construct a transversal S to γ at q_0. Construct a flow box on S; there is a point $q^* \in d$ in this flow box since q_0 is a limit point of d. Hence $p^* = \phi(t^*, q^*) \in S$ for some small t^*. However, $p^* \in \omega$ since the limit set is invariant and consequently S contains two points p^*, q_0 of ω. These points are distinct because $p^* \notin \gamma_0$ since otherwise we would infer that $q^* \in \gamma_0$ which contradicts our assumption that $q^* \in d$. This contradicts Lemma 7.4.4, showing that d must in fact be empty.

Remark: The second line of the proof may be expanded as follows. Take $q_k \in d$ with $q_k \to p$. Then $p \in \omega(\gamma) = d \cup \gamma_0$ since ω is closed. Suppose all such limit points p lay in d. Then d would be closed, and since d and γ_0 are disjoint closed sets, we could infer that ω is not connected.

With the aid of the preceding lemmas, we can now prove a rather general version of the Poincarè-Bendixson theorem. Theorem 7.4.1 can easily be inferred from:

Theorem 7.4.2 *(Poincarè-Bendixson Theorem) If γ is a bounded semi-orbit for which $\omega(\gamma)$ lies in Ω and contains no equilibrium points of f, then $\omega(\gamma)$ is a periodic orbit.*

Proof: Assume γ is not periodic, since otherwise the claim holds trivially. Since γ is a bounded semi-orbit, $\omega(\gamma)$ is non-empty. By assumption it contains no equilibrium points of f and consequently ω contains a non-constant orbit γ_0.

γ_0 is itself bounded, and hence we infer that $\omega(\gamma_0)$ is also bounded and non-empty. Let p_0 be a point in $\omega(\gamma_0) \subset \omega(\gamma)$. Since p_0 is a limit point of γ we can, by Lemma 7.4.2, construct a transversal S to γ containing p_0.

Recall that $\omega(\gamma)$ cannot meet S at more than one point and hence it meets S only at p_0. Since $\gamma_0 \subset \omega(\gamma)$, γ_0 meets S only at p_0. However, p_0 is a limit point of γ_0 and is a regular point for f so if γ_0 is not periodic then by Lemma 7.4.2 it intersects S infinitely often, the intersections approaching p_0 monotonically along S. It must therefore be the case that γ_0 is periodic, and by Lemma 7.4.5 we conclude that $\omega(\gamma) = \gamma_0$ is periodic. □

Orbits can of course be unbounded, but for bounded orbits the long-time behavior is, as remarked at the beginning of this section, "severely circumscribed," as the following theorem shows:

Theorem 7.4.3 *Let γ be a positive semiorbit contained in a compact subset K of Ω and suppose that K contains only a finite number of equilibrium points. Then*

1. *$\omega(\gamma)$ consists of a single point p which is a equilibrium point of f, and the orbital point $x(t) \to p$ as $t \to +\infty$, or*

2. *$\omega(\gamma)$ is a periodic orbit, or*

3. *$\omega(\gamma)$ consists of a finite set of equilibrium points together with their connecting orbits. Each such orbit approaches an equilibrium point as $t \to +\infty$ and as $t \to -\infty$*

Proof: The limit set $\omega(\gamma)$ may consist of regular points and of equilibrium points. We first consider the cases when one or the other of these is missing.

1. Suppose ω contains no regular points. Then it consists entirely of equilibrium points and, since it is connected, it must consist of exactly one of these, x_0 say. Inasmuch as $\omega = \{x_0\}$, the orbit approaches x_0 as $t \to \infty$.

2. Suppose next that ω contains a regular point but no equilibrium point. Then it contains a complete orbit and no equilibrium point, so it is periodic by the Poincarè-Bendixson theorem.

3. Finally suppose that ω contains both regular points and a (positive) number of equilibrium points, and in particular is therefore not a periodic orbit. Since it contains regular points it contains a complete orbit γ_0. The latter has an omega-limit set $\omega(\gamma_0)$. If the latter contained a regular point, then γ_0 and $\omega(\gamma_0)$ would have a point in common, by the argument used in the proof of the Poincarè-Bendixson theorem. Then γ_0 would be periodic, by Lemma 7.4.3, which would imply $\omega(\gamma)$ is periodic, by Lemma 7.4.5. It follows that γ_0 can have only equilibrium points as limit points and therefore only one such point, as in the reasoning of part (1) above. Thus each regular orbit approaches a unique equilibrium point as $t \to \infty$. Since each γ_0 is a full orbit, this argument can be repeated for $t \to -\infty$. $\Box$.

In part (3) of the preceding theorem, the equilibrium points x_+ and x_- to which an orbit γ_0 tends as $t \to +\infty$ and $t \to -\infty$ may be different, or they may be the same. In the following example, they are different.

Figure 7.2: The compact region K described in Example 7.4.1 is shown, together with a typical orbit. The three equilibrium points are denoted by dots. The value of the parameter has been taken to be $R = 0.2$. Any such orbit spirals outward, gradually becoming indistinguishable from the boundary, which is its ω-limit set.

Example 7.4.1 Consider the system

$$\dot{x}_1 = x_2 + x_1^2 - Rx_1\left(x_2 - 1 + 2x_1^2\right), \quad \dot{x}_2 = -2\left(1 + x_2\right)x_1, \tag{7.13}$$

where R is a parameter. The curves $x_2 = -1$ and $x_2 = 1 - 2x_1^2$ are invariant curves for this system (see Figure 7.2). The compact set K consisting of the finite region bounded by these curves together with its boundary contains three equilibrium points, at $(0,0), (1,-1)$ and $(-1,-1)$. Orbits starting in the interior of this region spiral outward toward its boundary, which is the ω-limit set for any such orbit. This ω-limit set contains two equilibrium points together with the two orbits connecting them, illustrating the third possibility enumerated in Theorem 7.4.3.

7.4.1 Limit Cycles

A comprehensive theory of periodic orbits in R^n and their stability is taken up in Chapter 9. However, when $n = 2$, we are able to draw a number of useful conclusions on the basis of the Poincarè-Bendixson theory and its supporting lemmas.

The setting that we wish to consider is the following. We Suppose that the region Ω contains an isolated periodic orbit Γ. We then show that nearby orbits spiral into Γ either as $t \to +\infty$ or as $t \to -\infty$.

Theorem 7.4.4 *Suppose that* Γ *is a non-constant, periodic orbit of the two-dimensional system (7.12) and there are no other periodic orbits within some neighborhood of* Γ*. Then every trajectory beginning sufficiently close to* Γ *spirals into it either as* $t \to +\infty$ *or as* $t \to -\infty$*.*

Proof: It is possible to choose a neighborhood N of Γ in which there are no equilibrium points, for otherwise there would be a sequence of equilibrium points $\{x_n\}$ for which $d(x_n, \Gamma) \to 0$ as $n \to \infty$. This sequence is bounded and therefore has a convergent subsequence, converging (say) to x_*. Since $f(x_n) = 0$, this is so for x_* as well, i.e., x_* is an equilibrium point. But $d(x_*, \Gamma) = 0$, i.e., $x_* \in \Gamma$: this contradicts the assertion that Γ is a non-constant, periodic orbit. Choose N so that it contains no equilibrium point of the system (7.12), and no periodic orbit other than Γ.

Establish a transversal S to Γ at a point p. Then Γ returns to p at intervals of time $\pm T$, where T is the (least) period of Γ. Orbits starting within a sufficiently small neighborhood of Γ must likewise intersect S within a time interval close to $\pm T$, by virtue of the continuity of solutions $x = \phi(t, x_0)$ with respect to initial data x_0. It therefore suffices to consider orbits starting on S within a sufficiently small neighborhood of Γ (smaller than the neighborhood N of the preceding paragraph). Let q_0 be such a point for some orbit γ and let q_1 be the next intersection of γ with S for t increasing, q_{-1} the next intersection for t decreasing. These two points cannot be equal or γ would be periodic, which is excluded. Therefore one of them is closer to p than the other; for definiteness, say $d(q_1, p) < d(q_{-1}, p)$. We consider the Bendixson pocket B_1 formed by following γ from q_0 to q_1 and then S from q_1 back to q_0, and we repeat this, getting a second intersection q_2. It is clear that $d(q_2, p) < d(q_1, p)$, and so on: successive intersections are nearer to p.

Suppose the sequence $\{q_n\}$ *failed* to converge to p. Denote the interior of the curve Γ by I_Γ and consider the closure of the set B_1/I_Γ. The positive semi-orbit γ is confined to this compact set, which is free of equilibrium points and which then, by the Poincarè-Bendixson theorem, contains a periodic orbit – a contradiction. Therefore the intersections $\{q_n\}$ converge to p and the orbit γ spirals into Γ as $t \to \infty$. Had we assumed that $d(q_{-1}, p) < d(q_1, p)$ instead, then we would conclude that γ spirals into Γ as $t \to -\infty$. □

An isolated periodic orbit like that of this theorem is called a *limit cycle* if all sufficiently nearby orbits are attracted to it as $t \to +\infty$ or as $t \to -\infty$ (cf Problem 4 of Exercises 7.4.1) . However, the theorem allows for the 'peculiar' possibility that orbits outside Γ tend to Γ as $t \to +\infty$ whereas those inside tend away from Γ in this limit (or tend toward Γ as $t \to -\infty$) (cf. Problem 5 of Exercises 7.4.1). It will be a consequence of the general theory of Chapter 9 that this peculiar possibility is excluded if the vector field is assumed to be C^1.

We consider next an important class of cases for which periodic orbits are *not* isolated.

7.4.2 Hamiltonian systems

Let the system (7.12) be in canonical Hamiltonian form, defined as follows:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \ \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{7.14}$$

where we have written $x = (q,p)$ to conform to standard terminology in Hamiltonian dynamics, and $H(q,p) = H(x)$ is a smooth function in a domain Ω in R^2. Suppose the domain Ω contains an isolated equilibrium point $x_0 = (q_0, p_0)$ which is, moreover *stable*: given a neighborhood U_0 of x_0, all orbits starting in a sufficiently small neighborhood $V_0 \subset U_0$ remain in U_0 for all $t > 0$. This will be true if the function $H(x) - H(x_0)$ is positive-definite near x_0.[4] We may assume U_0 is small enough to exclude any equilibrium points other than x_0. We may then infer, under fairly general assumptions, that all orbits starting in V_0 are periodic. The precise result, stated for the case when the equilibrium point in question is located at the origin of coordinates, is as follows.

Proposition 7.4.1 *Suppose that the system (7.14) has a stable equilibrium point at the origin and that the domain Ω contains the origin and no other equilibrium point. Suppose there is a direction $\hat{t}$ such that*

$$\hat{t} \cdot \nabla H > 0 \tag{7.15}$$

on $\Omega \setminus \{O\}$. Then there is a subdomain V_0 of Ω such that all orbits beginning in $V_0 \setminus \{O\}$ are periodic.

Remark: By the notation $\Omega \setminus \{O\}$ we mean the punctured region consisting of Ω with the origin removed.

[4] We discuss stability more fully in the next chapter.

Proof: The part of the line segment $x = s\hat{t}$ that lies in $\Omega \setminus \{O\}$ is easily seen to be a transversal S to orbits in this region. Choose V_0 so that any orbit beginning in V_0 remains in Ω for all subsequent times. Let γ be any orbit beginninig in $V_0 \setminus \{O\}$; it is necessarily nonconstant. If it is not periodic it must, by Lemma 7.4.2, intersect S successively at distinct points $x_1 = x(t_1), x_2 = x(t_2), \ldots$. Because of the condition (7.15), $H(x_1) \neq H(x_2)$. But the structure of the Hamiltonian system is such that $H(x(t))$ is constant on orbits. It follows that γ is periodic. □

The region of periodic orbits need not be small. Given a stable equilibrium point, there is always an invariant domain containing it. If it can be ascertained that it satisfies the conditions for Theorem 7.4.2 to hold, we can infer that all orbits are periodic in that domain.

Example 7.4.2 Let

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{q^3}{3}.$$

This Hamiltonian has a separatrix S passing through the unstable equilibrium point at $(q,p) = (-1,0)$ (see Figure 7.3). This curve, on which $H(q,p) = 1/6$, separates the plane into regions of different behavior. It is also an orbit, which approaches the (unstable!) $(q,p) = (-1,0)$ both as $t \to +\infty$ and as $t \to -\infty$. The region marked A in Figure 7.3, which contains the stable equilibrium point $(q,p) = (0,0)$, is invariant and bounded. Any orbit in this region satisfies the conditions of Theorem 7.4.2, so all orbits in A are periodic.

PROBLEM SET 7.4.1

1. Consider a two-dimensional system $\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$ and consider an invariant curve of this system of the form $y = h(x)$, where h is a differentiable function on some interval I of the x axis. Show that a necessary and sufficient condition for h to represent such a curve is that

$$h'(x) f(x, h(x)) = g(x, h(x)) \tag{7.16}$$

on I.

Figure 7.3: A phase-space diagram for the system with the Hamiltonian of Example 7.4.2. The region marked A consists of the stable equilibrium point at the origin together with periodic orbits surrounding it. All other orbits either lie on the separatrix S (and tend toward $(-1, 0)$ as $t \to \pm\infty$) or are unbounded.

2. For Example 7.4.1 of the text, verify, using the criterion (7.16) above, that the curves $x_2 = -1$ and $x_2 = 1 - 2x_1^2$ are indeed invariant.

3. Put $R = 0$ in Example 7.4.1. Show that the system is then Hamiltonian. Find the Hamiltonian function.

4. Consider the system

$$\dot{x} = (1 - \sqrt{x^2 + y^2})x - y, \quad \dot{y} = (1 - \sqrt{x^2 + y^2})y + x.$$

   Identify any equilibrium points and periodic orbits and, in the case of the latter, determine whether nearby orbits spiral toward or away from the periodic orbit.
   Hint: rewrite this system in terms of polar coordinates in the plane.

5. Repeat the previous problem for the system

$$\dot{x} = |1 - \sqrt{x^2 + y^2}|x - y, \quad \dot{y} = |1 - \sqrt{x^2 + y^2}|y + x$$

   where the vertical bars $|\cdot|$ denote absolute value.

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