

Chapter 1

Introduction and first-order equations

In this introductory chapter we define ordinary differential equations, give examples showing how they are used and show how to find solutions of some differential equations of the *first order*.

1.1 What is an ordinary differential equation?

A differential equation is one relating a function and its derivatives, and the objective is to find a function satisfying the given equation. An example may help to fix ideas.

Example 1.1.1 Let x be an independent variable on an interval I of the axis of real numbers, and consider the equation

$$\frac{u'}{u} - 2 = 0, \tag{1.1}$$

where u is an unspecified function of x and the prime indicates differentiation: $u' = du/dx$. This equation can also be written in the alternative form

$$u' = 2u. \tag{1.2}$$

The function $u = f(x) = C \exp(2x)$ satisfies this differential equation for any choice of the constant C . This can be verified by forming the derivative $f'(x) = 2C \exp(2x)$ appearing on the left-hand side of the alternative form,

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and the the combination $2f(x) = 2C \exp(2x)$ appearing on the right-hand side, and checking that they are indeed equal for each value of x . \square

The equation (1.1) is a differential equation. The function $f(x) = C \exp(2x)$ satisfying it will be referred to as a *solution* of the given differential equation. The alternative form of the equation, equation (1.2), in which the derivative is expressed in terms of the function, is the *standard* form of the equation.

A more general version of a differential equation is the following. Suppose you're given, for some function $\phi(y, z)$ of two variables, the equation

$$\phi(u(x), u'(x)) = 0, \quad (1.3)$$

with the objective of finding a function $y = u(x)$ on an interval of the x axis. This is the problem of finding a function $u(x)$ of the variable x (the *independent* variable) such that equation (1.3) becomes an identity on some interval of the x -axis. In the preceding example, the function ϕ took the form

$$\phi(y, z) = \frac{z}{y} - 2$$

and the solution of the differential equation was given as $u = C \exp(2x)$. In the more general case of equation (1.3) it may not be possible to give a simple formula representing the solution. In fact, the problem arises whether there is *any* function $u(x)$ such that $\phi(u(x), u'(x))$ vanishes identically on an interval of the x axis. This problem, and related problems, will be discussed later in this book. These issues are most conveniently discussed for differential equations written in standard form and most of the general results of the theory of ordinary differential equations are given for equations in this form.

Rewriting the differential equation in standard form was a simple matter in example (1.1.1). In the more general case of equation (1.3) this requires solving the functional equation $\phi(y, z) = 0$ for z as a function of y , say $z = g(y)$. If this can be achieved then the substitutions $y = u, z = u'$ achieve the standard form

$$u' = g(u). \quad (1.4)$$

Equation (1.4) is actually a restricted version of the standard form; the more general version is given below (equation 1.11).

The goals of this book are not only to establish conditions on the function g so that equations like (1.4) and (1.11) possess solutions, but also to develop techniques for finding or describing these solutions.

What's "ordinary" about an ordinary differential equation? The qualifier "ordinary" is used to indicate differential equations having one independent variable in contrast to those having two or more independent variables; the latter are called *partial differential equations*. An example of the latter is

$$\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0. \quad (1.5)$$

Here the object is to find a solution function $u(s, t)$ depending on the two independent variables s and t . It therefore involves partial derivatives, whereas an ordinary differential equation involves only ordinary derivatives. We will not investigate partial differential equations in any depth in this book. There are, however, instances wherein ordinary differential equations play important roles in the study of partial differential equations, and we will mention some of the latter to motivate our study of the associated ordinary differential equations (cf. Problem Set 1.2.1).

Since we'll be concerned mostly with ordinary rather than partial differential equations, we'll often drop the qualifier "ordinary" in this book and use the term "differential equation" to mean "ordinary differential equation" unless the contrary is explicitly stated.

We conclude this section with a number of examples of differential equations of the form of (1.3) and (1.4).

Example 1.1.2 Suppose the function is $\phi(y, z) = y^2 + z^2 - 1$. The differential equation for $y = u(x)$ is

$$u^2 + u'^2 - 1 = 0.$$

In this case, it is easy to verify that $u(x) = \sin(x + c)$, for any choice of the constant c , is a solution. The standard form of the differential equation is either $y' = \sqrt{1 - y^2}$ or $y' = -\sqrt{1 - y^2}$. \square

Example 1.1.3 According to the laws of mechanics, a particle of mass m moving in the gravitational field of the earth has a constant energy

$$E = (1/2)mv^2 - mMG/r$$

where v is its speed, G is the gravitational constant, M is the mass of the earth, and r is the distance of the particle from the center of the earth; if R

is the radius of the earth, $r \geq R$. If the particle is moving radially outward, then $v = dr/dt > 0$ where t represents time, and the position of the particle is governed by the differential equation (obtained by solving for v in the energy equation above)

$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} + \frac{2MG}{r}}. \quad (1.6)$$

The constant value of the energy E can be determined from the initial values of the position and speed. Here the differential equation for r is in the standard form of equation (1.4) with $g(r) = \sqrt{2E/m + 2MG/r}$. \square

In this example we have used t as the independent variable instead of x , since it represents the time. The designation of variables is of course arbitrary, but some usages are traditional, and we shall adhere to them. It is also traditional¹ to use a dot to indicate a derivative with respect to time ($\dot{x} \equiv dx/dt$).

Example 1.1.4 Suppose the speed \dot{x} of a particle on the x axis is controlled so that $x\dot{x} = 1$. Then, solving the functional problem, we find

$$\frac{dx}{dt} = \frac{1}{x}.$$

This can be solved by noticing that the original form of the equation implies that

$$\frac{d}{dt} \frac{x^2}{2} = 1.$$

If $x(0)$ is the starting point of the motion when $t = 0$, then integrating each side of the equation gives

$$x(t)^2 = x(0)^2 + 2t.$$

The solution is then found by taking a square root. \square

Example 1.1.5 If you have money deposited in an interest-paying account, it will grow with time if you do not make any withdrawals. How fast it grows depends on the annual interest rate I and the frequency of compounding. If the interest rate is 5% and it is only compounded annually, at the end of one

¹This goes back to the beginnings of calculus.

year you will have $1.05P_0$ on deposit, where P_0 is your initial deposit. At the end of two years you will have $(1.05)^2P_0$, etc. If it is compounded quarterly, at the end of one quarter your deposit will have increased to $(1 + 0.05/4)P_0$ and after one year to $(1 + 0.05/4)^4P_0 \approx 1.0509P_0$. If it is compounded monthly the initial deposit gets increased by a factor of $(1 + 0.05/12)$ at the end of each month, so that by the end of the year it has increased by the factor

$$(1 + 0.05/12)^{12} \approx 1.0512. \quad (1.7)$$

In general, if an annual interest I (expressed as a decimal: $I = 0.05$ in the examples above) is compounded in N equal intervals, the amount P_1 on deposit after one of these intervals is

$$P_1 = (1 + I/N)P_0. \quad (1.8)$$

More generally, if $P(t)$ is the principal at time t , then after an interval of length $1/N$ the principal will be

$$P(t + 1/N) = (1 + I/N)P(t) = P(t) + (I/N)P(t).$$

Suppose your bank compounds *continuously*. This means that if your balance at time t is $P(t)$ and Δt is a sufficiently small interval of time, then

$$P(t + \Delta t) \approx P(t) + I\Delta tP(t),$$

the approximation improving as Δt tends to zero; there is an implicit assumption in this formula that t is measured in years so that if the time interval is one day, $\Delta t = 1/365$. From this formula one deduces that the balance $P(t)$ on deposit at any time satisfies the differential equation

$$\frac{dP}{dt} = IP. \quad \square \quad (1.9)$$

1.1.1 The general, first-order case

There is a more general form of differential equation than that of equation (1.3). Suppose $\phi = \phi(t, x, y)$ is a prescribed function of three variables and the problem is that of finding a function $x(t)$ such that

$$\phi(t, x(t), \dot{x}(t)) = 0 \quad (1.10)$$

on an interval of the t axis. The standard form of this equation would be

$$\frac{dx}{dt} = f(t, x) \quad (1.11)$$

where the function f is obtained by solving the equation $\phi(t, x, y) = 0$ for y as a function of t and x .

There are rules whereby one can determine whether the equation (1.10) can be solved and put into the standard form (1.11). Suppose $\phi(t, x, y)$ is continuously differentiable (C^1 for short) in a domain D of the txy -space. Suppose further that the following two conditions hold:

$$\phi(t_0, x_0, y_0) = 0 \text{ for some } (t_0, x_0, y_0) \in D$$

and

$$\frac{\partial \phi}{\partial y}(t_0, x_0, y_0) \neq 0.$$

Then, according to the implicit-function theorem (see, for example, [7]), the equation $\phi(t, x, y) = 0$ possesses a solution $y = f(t, x)$ in a neighborhood N of (t_0, x_0) in the tx -space. This solution reduces to y_0 when $(t, x) = (t_0, x_0)$. Moreover, if N is sufficiently small, it is the *unique* solution reducing to y_0 when $(t, x) = (t_0, x_0)$. This function f is C^1 on N .

We have noted in the context of equation (1.3) above that the question arises whether there is in fact *any* solution. For a general ordinary differential equation in one of the forms (1.10) or (1.11), this is the question whether there is a function $x(t)$ on an interval of the real t axis which turns the equation into an identity on that interval. This general *existence* question will be taken up later on (cf. §1.4 and Chapter 6 below). For the time being we will be concerned with simple problems for which we can express the solutions explicitly in terms of known functions and their integrals; for such problems the issue of existence is clearly settled in the affirmative.

By way of example, note that there is one particularly simple class of differential equations for which the solution is immediate. Suppose the function f in equation (1.11) does not depend on x : $f = f(t)$. Then the fundamental theorem of calculus shows that the solution is

$$x(t) = \int^t f(s) ds + \text{constant}. \quad (1.12)$$

Thus to obtain the solution one need only integrate a known function. Such a solution involving the integration of a known function is sometimes referred to as a solution by *quadrature*.²

1.1.2 The initial-value problem

The solution by quadrature includes an arbitrary constant and is therefore not just one solution, but a one-parameter family of solutions, one for each value of the constant; this was likewise true of example 1.1.4 above. We can only have a unique solution of the differential equation if we somehow specify this constant. One way to do this is to require that the function $x(t)$ not only satisfy the differential equation but also satisfy the condition that it take on a prescribed value x_0 at a prescribed point t_0 . This lack of uniqueness turns out to be characteristic of more general differential equations as well, and the remedy suggested above to render the solution unique turns out to be appropriate in a variety of naturally occurring circumstances. For this reason we define the *initial-value problem*:

$$\frac{dx}{dt} = f(t, x), \quad x|_{t=t_0} = x_0. \quad (1.13)$$

Here the function f and the *initial data* t_0, x_0 are prescribed.

Example 1.1.6 Suppose that the equation is

$$\dot{x} = t,$$

and the initial data are:

$$x = 3 \text{ when } t = 1.$$

The solution of the equation is

$$x = (1/2)t^2 + c,$$

and the imposition of the initial data implies that $3 = 1/2 + c$ or $c = 5/2$. The solution of the initial-value problem is therefore

$$x = (1/2)t^2 + 5/2. \quad \square$$

²This is a quaint term from an earlier era; it has multiple meanings. Here we'll use it to mean an integral of a function that may be regarded as known, whether explicitly in terms of elementary functions, numerically, or by other means.

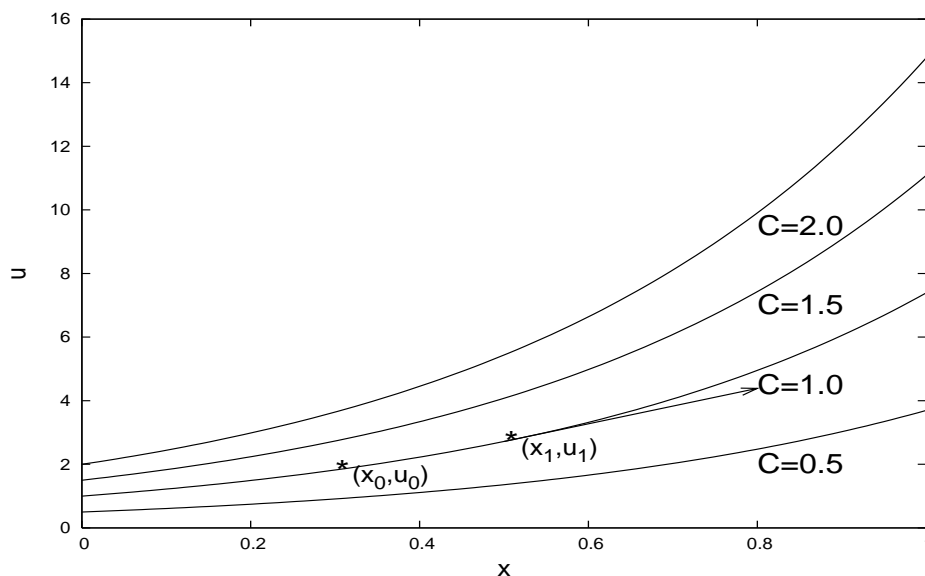


Figure 1.1: A family of solution curves for the equation $u' = g(u)$ is shown, in this case with $g(u) = 2u$ as in equation (1.2). The solutions $u = C \exp(2x)$ represent curves in the x, u plane. They are shown for several choices of the constant C . If initial data $u(x_0) = u_0$ are specified, the curve must pass through the point (x_0, u_0) : this effects a choice of the constant C . Any choice of a point in the x, u plane – say (x_1, u_1) – produces a slope $g(u)$ which the solution through that point must have. The arrow indicates the direction of a line having that slope.

There is a geometric picture associated with the initial-value problem. The essence of it consists of viewing the solution of a differential equation as a curve in space. The initial condition that $x|_{t=t_0} = x_0$ represents the requirement that the curve pass through a specified point (t_0, x_0) . It can be illustrated by drawing a family of solutions as in Figure 1.1 and noting that only one of them passes through a specified point (x_0, u_0) in the notation of the diagram.

A closely related geometric picture is based on the slopes of these curves. If $x(t)$ is a solution of equation (1.13) then $\dot{x}(t) = f(t, x(t))$, and therefore at any point $(t_1, x(t_1))$ of its graph the slope is $f(t_1, x(t_1))$. Conversely, if (t_1, x_1) is a point in the tx -plane then $f(t_1, x_1)$ is the slope of the tangent line of the graph of any solution passing through the point (t_1, x_1) . This leads to the notion of a *direction field*. This is a family of vectors. The vector whose tail lies at the point (t, x) has the components $(h, f(t, x)h)$ for some convenient

choice of the constant h , and represents the tangent to any solution curve passing through that point. An example is shown in Figure 1.1.

1.1.3 Higher-order equations

The examples given above are of first-order differential equations, i.e., those in which only the first derivative of the dependent variable occurs. These are special, and especially simple, and, beginning with Chapter 2, you will encounter more general differential equations like that of the following example.

Example 1.1.7 A particle of mass m moving on the x -axis under the influence of a force $f(x)$ depending on its position on the axis is governed by *Newton's Second Law of Motion*:

$$m\ddot{x} = f(x),$$

where $\ddot{x} \equiv d^2x/dt^2$ is shorthand notation for the second time derivative. \square

This is an example of a *second-order* differential equation since the highest-order derivative occurring in it is the second. In applications differential equations of arbitrarily high order can occur. For a positive integer n , the general n^{th} -order differential equation has the form

$$\phi(t, x, dx/dt, \dots, d^n x/dt^n) = 0.$$

It is put into standard form by solving this equation for the highest derivative:

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right). \quad (1.14)$$

It is traditional to begin discussions of the theory of ordinary differential equations with those of the first order. This introduces the student to many features of the theory in a relatively uncluttered context, so we shall adhere to this sensible tradition in the remainder of this chapter. In the examples above we have assumed that the number system is the *real* number system, and accordingly that all functions are real-valued. We shall continue to assume this unless there is an explicit statement otherwise.

1.2 Linear Equations

A function of one variable $f(x)$ is linear if, for arbitrary numbers α, x ,

$$f(\alpha x) = \alpha f(x). \quad (1.15)$$

It is easy to see from this definition that any linear function has the form $f(x) = kx$ for some constant k and that any such function is linear. A function $f(t, x)$ of two variables is then linear in the second (x) variable if and only if $f(t, x) = k(t)x$, where k can be an arbitrary function of t . A linear differential equation is one which, when written in the standard form (1.11), has a linear function $f(t, x)$ on the right-hand side:

$$\frac{dx}{dt} = k(t)x. \quad (1.16)$$

Unless otherwise indicated we'll assume that the function k is continuous.

1.2.1 The Linear, Homogeneous Equation

The equation (1.16) will be referred to as a *linear, homogeneous* equation, for reasons that will soon become clear. It's obvious by substitution that the function $x(t) \equiv 0$ is always a solution of the linear, homogeneous equation. However, if at some point $t = t_0$ $x(t_0) \neq 0$, then one can, for values of t near t_0 , divide through in equation (1.16) by x . Then

$$\frac{1}{x} \frac{dx}{dt} = \frac{d}{dt} \ln x = k(t).$$

From this we easily infer, by integrating from t_0 to t and solving for $x(t)$, that the solution is

$$x(t) = x(t_0) \exp \left\{ \int_{t_0}^t k(s) ds \right\}. \quad (1.17)$$

This formula gives the general solution to the linear, homogeneous equation of first order. It agrees with the previous observation that $x \equiv 0$ is a solution: if $x(t_0) = 0$, this formula implies that $x(t) \equiv 0$. On the other hand, if $x(t_0) \neq 0$, it implies that $x(t) \neq 0$ for any value of t .³

³We assume that the function k is bounded, since otherwise this conclusion can fail.

Many so-called *rate* problems take the form of equation (1.16) where $k(t) = k$ is constant. For these the solution is immediate:

$$x(t) = x(t_0) e^{k(t-t_0)}.$$

A case in point is equation (1.9) of Example 1.1.5. Another example is the following.

Example 1.2.1 Let $x(t)$ represent the number of individuals in an isolated population. We ignore the fact that x should be an integer and assume it can be viewed as a smooth function of time, a plausible assumption if the population is large. The population could be a human, insect or some other biological group. That it is isolated means we can ignore migration, inwards or outwards. The simplest model of population change then assumes that there is a constant birth rate b and a constant death rate d . Then the increase in x attributable to births during a short time-interval Δt is approximately $x(t) b \Delta t$, and the decrease in x , attributable to deaths, is $x(t) d \Delta t$. The net change is therefore $x(t + \Delta t) - x(t) \approx (b - d) x(t) \Delta t$. As in the derivation of equation (1.9) above we infer that the population is governed by the equation

$$\dot{x} = kx \tag{1.18}$$

where $k = b - d$ and reflects the difference between the birth and death rates. The solution is

$$x(t) = x_0 e^{k(t-t_0)}.$$

If births exceed deaths, $k > 0$ and the population increases exponentially; if deaths exceed births, $k < 0$ and the population decreases exponentially. \square

In the previous example, if $k < 0$ so that the population decays, it never decays to zero: the exponential function is always positive. The population would approach zero as $t \rightarrow \infty$, but would never reach it in finite time. Likewise, if $k > 0$, the population would become arbitrarily large as $t \rightarrow \infty$, but would not become infinite in finite time. Similar remarks hold for rate problems with variable rate.

Example 1.2.2 The initial-value problem

$$\frac{dx}{dt} = -tx, \quad x|_{t=0} = 1 \tag{1.19}$$

has the solution $x = \exp\{-t^2/2\}$ according to the formula (1.17). \square

1.2.2 The Inhomogeneous Equation

The equation

$$\frac{dx}{dt} = k(t)x + a(t), \quad (1.20)$$

where a is a specified function of t , differs from the linear equation by the addition to the right-hand side of a term independent of x . We'll assume that a is continuous unless the contrary is specified. The resulting equation, called the *linear inhomogeneous* equation, may be solved in closed form as follows. Multiply each side by

$$K(t) = \exp \left\{ - \int_{t_0}^t k(s) ds \right\}, \quad (1.21)$$

where t_0 is an appropriate initial point. An easy calculation shows that equation (1.20) may be rewritten

$$\frac{d}{dt} (Kx) = Ka,$$

so, integrating above from t_0 to t and then multiplying each side by

$$K(t)^{-1} = \exp \left\{ + \int_{t_0}^t k(s) ds \right\},$$

we find

$$x(t) = \exp \left\{ \int_{t_0}^t k(s) ds \right\} \left[x(t_0) + \int_{t_0}^t \exp \left\{ - \int_{t_0}^s k(u) du \right\} a(s) ds \right], \quad (1.22)$$

where we have used the fact that $K(t_0) = 1$. This can be rewritten as

$$x(t) = \exp \left\{ \int_{t_0}^t k(s) ds \right\} x(t_0) + \int_{t_0}^t \exp \left\{ \int_s^t k(u) du \right\} a(s) ds. \quad (1.23)$$

This somewhat cumbersome formula provides the complete solution to the linear, inhomogeneous differential equation of the first order. The exact representation of the solution of a differential equation in literal terms, reducing its solution to quadratures as in equation (1.23), is rare. We'll make repeated use of this formula in this book in order to exploit this stroke of good luck.

Example 1.2.3 The initial-value problem

$$\dot{x} = tx + t, \quad x(0) = 1$$

has, according to the formula (1.23), the solution

$$x(t) = \exp(t^2/2) + \exp(t^2/2) \left[-\exp(-t^2/2) + 1 \right] = 2 \exp(t^2/2) - 1.$$

1.2.3 Duhamel's principle

The general solution (1.23) to the linear, inhomogeneous initial-value problem consists of two terms: the first is the same as the solution of the homogeneous problem as given by equation (1.17) and does not involve the inhomogeneous term $a(t)$, and the second is a so-called *particular integral* and is itself a linear expression in $a(t)$. This representation of the solution as the sum of a solution of the homogeneous problem and a particular integral is characteristic of linear initial-value problems.

In this representation, it is easy to verify that a complete solution of the initial-value problem can be obtained once *any* particular integral has been found. Suppose then that the homogeneous problem may be regarded as solved. Duhamel's principle enables one to solve the inhomogeneous problem by providing an expression for a particular integral.

Proposition 1.2.1 (*Duhamel's principle*) *Let $s \geq t_0$ be prescribed and denote by $X(t, s)$ the solution of the initial-value problem*

$$\frac{dX}{dt} = k(t)X, \quad X|_{t=s} = 1. \quad (1.24)$$

Then a particular integral P of the inhomogeneous equation (1.20) is given by the formula

$$P(t) = \int_{t_0}^t X(t, s)a(s) ds. \quad (1.25)$$

The proof is a verification depending only on the formula for differentiating an integral containing the independent variable both in the integrand and in the limit of integration (see Problem 20 below). In the present case $X(t, s)$ can be found explicitly:

$$X(t, s) = \exp\left\{\int_s^t k(u) du\right\}.$$

Thus Duhamel's principle simply recovers the second term in equation (1.23).

It may seem that little has been achieved: a known result has been recovered via a procedure that appears to have been "drawn from thin air." But first, Duhamel's principle has applications beyond the present simple example, and we shall invoke it later in this text. Second, it is possible to motivate it, beginning with the linearity of the system, so that it no longer

appears to be “drawn from thin air.” Here we mention only a general feature of the linearity that helps to motivate it, and we leave the rest to the resourcefulness of the reader.

Consider the initial-value problems

$$\frac{dx_j}{dt} = k(t)x_j + a_j(t), \quad x_j(t_0) = 0 \quad (1.26)$$

for a set of values $j = 1, 2, \dots, n$. If $a(t) = \sum_j a_j(t)$ then the solution of the initial-value problem

$$\frac{dx}{dt} = k(t)x + a(t), \quad x(t_0) = 0 \quad (1.27)$$

is $x(t) = \sum_j x_j(t)$.

PROBLEM SET 1.2.1

In problems 1-3, put the differential equations $\phi(t, x, \dot{x}) = 0$ into standard form:

1. $\phi(t, x, \dot{x}) = (1 + t^2)\dot{x} + 1 - x^2$.
2. $\phi(t, x, \dot{x}) = 1 - \dot{x}^3$.
3. $\phi(t, x, \dot{x}) = x^2 + \dot{x}^2 - t^2$.
4. In Example 1.2.1 above, suppose that in addition to the change in the population due to births and deaths, there is immigration at the constant rate $I > 0$ per unit time. Derive the generalization of equation (1.18).

Solve the following four initial-value problems:

5. $\dot{x} = t^2 - e^t$, $x(0) = 2$.
6. $\dot{x} = 0.05x$, $x(0) = 100$. This is the result of “continuous compounding” as described in Example 1.1.5 above. Evaluate the solution numerically at $t = 1$, i.e., at the end of one year, and compare this with monthly compounding as given in equation (1.7).
7. $\dot{x} = -\alpha x + \gamma$, $x(0) = 1$, where α and γ are positive constants.
8. $\dot{x} = -tx + 1$, $x(1) = 0$.

9. Suppose that $x_1(t)$ and $x_2(t)$ are both solutions of equation (1.16). Show that $x(t) = c_1x_1(t) + c_2x_2(t)$ is then also a solution, for arbitrary values of the constants c_1 and c_2 .
10. An *equilibrium* solution is one that is constant, i.e. does not depend on the independent variable. Therefore if \tilde{x} is such a solution, $\dot{\tilde{x}} = 0$ for all t . Find an equilibrium solution \tilde{x} of the equation of Problem 7 above. Show that, if $x(t)$ is *any* solution of this equation, $x(t) \rightarrow \tilde{x}$ as $t \rightarrow +\infty$.

The following three problems involve partial-differential equations:

11. Refer to the partial-differential equation (1.5). Let f be *any* differentiable function, and verify that $u(s, t) = f(s - t)$ is a solution.
12. For the partial-differential equation

$$\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = \alpha u,$$

where α is a constant, show that one possible solution is $f(s + t)$, where f satisfies an ordinary differential equation. Find this solution.

13. For the partial-differential equation

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \alpha u,$$

where α is a constant, introduce polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$. With $u(x, y) = v(r, \varphi)$, deduce that the equation takes the form

$$r \frac{\partial v}{\partial r} = \alpha v.$$

Give the general solution of this equation, noting that it involves, not an arbitrary constant, but an arbitrary function.

In the following four problems, the first-order equation (1.20) is assumed to have *periodic* coefficients: $k(t + T) = k(t)$ and $a(t + T) = a(t)$ for all t . Here T is a constant, the least period of the coefficients k and a .

14. Let $a(t) = 0$ and $k(t) = \cos t$ so that $T = 2\pi$. Find the solution with initial value $x(0) = x_0$ at $t = 0$. Is the solution periodic?
15. Consider the preceding problem ($a(t) = 0$) except that $k(t) = (\cos t)^2$. What is the least period of the function k ? Is the solution periodic?

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16. Returning to the general case in which k and a are periodic with least period T , formulate a general condition on these two functions and on the initial value $x(0) = x_0$ implying that $x(T) = x_0$.
17. Apply the result of the preceding problem to the case

$$k \equiv 1, \quad a(t) = \sin t, \quad (T = 2\pi)$$

to find an initial value $x(0) = x_0$ such that $x(T) = x_0$, and verify that the solution is periodic in this case and only in this case.

18. The formula (1.23) for the solution of the inhomogeneous equation (1.20) continues to be valid when the coefficient $a(t)$ is piecewise continuous. Use it to solve the initial-value problem

$$\dot{x} = 2x + a(t), \quad x(0) = 4, \quad \text{where } a(t) = \begin{cases} -1 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1. \end{cases}$$

Show that $x(t)$ is continuous everywhere but \dot{x} fails to be continuous at $t = 1$.

19. Refer to example (1.1.5). Recall from calculus the result

$$\lim_{k \rightarrow \infty} (1 + x/k)^k = e^x.$$

From equation (1.8) obtain the result

$$P_M = (1 + I/N)^M P_0$$

for the principal after M intervals of time each of length $h = 1/N$. If time t has elapsed, then $Mh = t$. Use these statements along with the limit formula above to find the limit of P_M as $M \rightarrow \infty$, and therewith an alternative way of deriving the formula for continuously compounded interest.

20. Given

$$f(t) = \int_{t_0}^t g(t, s) ds$$

where $g(t, s)$ is a function that has continuous partial derivatives, show that

$$f'(t) = g(t, t) + \int_{t_0}^t \frac{\partial g}{\partial t}(t, s) ds$$

and complete the derivation of Duhamel's principle (1.2.1 above).

1.3 Other solvable equations

There are certain other classes of first-order equations that can be reduced to quadratures. These differential equations represent rather special cases but they do arise in a variety of circumstances and the methods described below are then useful.

1.3.1 Separable Equations

One of these classes of differential equations, called *separable*, has the form

$$y' = f(x)g(y). \quad (1.28)$$

It can be reduced to quadratures in the form

$$\int_{y_0}^y \frac{du}{g(u)} = \int_{x_0}^x f(v) dv \quad (1.29)$$

where x_0 and y_0 are appropriate constants; if the equation (1.28) is provided with the initial data $y(x_0) = y_0$, the initial-value problem is solved by the formula (1.29) in an interval of the x -axis containing x_0 provided $g(y_0) \neq 0$. The linear, homogeneous equation (1.16) falls into this category, but so do many others. The quadratures appearing in equation (1.29) do not complete the solution: one still needs to solve the remaining equation for y as a function of x . This can always be done in principle: Problem 2 of the following problem set addresses this issue. However, obtaining an explicit expression for the solution, in terms of known functions, may not be possible. This is the case for the following example, and is further emphasized by the result of Problem 3 of Problem Set 1.3.1.

Example 1.3.1 The differential equation

$$\dot{x} = \sqrt{(\alpha^2 - x^2)(\beta^2 - x^2)},$$

for appropriate values of the constants α, β , arises in mechanics in the study of the motion of a rigid body. It is separable, but the quadrature

$$\int \frac{dx}{\sqrt{(\alpha^2 - x^2)(\beta^2 - x^2)}}$$

is an *elliptic integral* and the solutions are expressed in *elliptic functions*, which represent a new class of functions, effectively invented to deal with this and related examples⁴ □.

The following example, on the other hand, *can* be integrated in elementary functions (cf. Problem 7 of Problem Set 1.3.1 below).

Example 1.3.2 The *logistic* equation

$$\dot{x} = kx(1 - x/N) \quad (1.30)$$

describes population growth limited by competition for resources. The constant k reflects the difference between the birth and death rates ($k > 0$ for a growing population) and N is the “carrying capacity” of the environment. This equation is nonlinear, but it is separable. If $x(0) = x_0$ expresses the initial population, then

$$\int_{x_0}^x \frac{du}{u(1 - u/N)} = kt. \quad \square$$

Example 1.3.3 Imagine an object dropped from a great height with initial velocity $v = 0$. Measure distance and velocity downward toward the surface of the earth. Then the equation governing the velocity of the object may be written

$$m\dot{v} = mg - Cv^2 \quad (1.31)$$

where m is the mass of the object, g is the acceleration of gravity (regarded as a constant) and the second term reflects the slowing effect of air resistance. C is a positive constant depending on the nature of the object. This equation is separable. □

Homogeneous equations

A class of differential equations in the standard form (1.11) that can be reduced to separable form is that for which the right-hand side is *homogeneous of degree zero*. A function $f(x, y)$ is said to be *homogeneous of degree k* if

$$f(ax, ay) = a^k f(x, y) \text{ for arbitrary values of } a. \quad (1.32)$$

⁴There is an extensive literature on the subject of these functions; see for example [1].

For example, the function $f(x) = kx$ for constant k is homogeneous of degree one (and that's why the linear equation 1.16 is also referred to as homogeneous). A function that is homogeneous of degree zero has the form $f(x, y) = g(y/x)$, as can be verified by choosing $a = 1/x$ in the definition of homogeneity. The differential equation $y' = g(y/x)$ can then be put into separable form (1.28) by the substitution $y = xu$: $u' = x^{-1}(g(u) - u)$.

Example 1.3.4 Consider the initial-value problem

$$y' = \frac{y^2}{2x^2} + \frac{1}{2}, \quad y(1) = 2.$$

The substitution $y = xu$ leads to the equation $xu' = \frac{1}{2}(u - 1)^2$ or

$$\int^u \frac{dv}{(1-v)^2} = (1-u)^{-1} = \frac{1}{2} \ln x + c.$$

The initial data, which imply that $u = 2$ when $x = 1$ imply that $c = -1$. This gives u and then

$$y = xu = x - \frac{x}{(1/2) \ln x - 1}.$$

□

1.3.2 Exact Equations

Both linear and separable equations discussed above are examples of a more general category, that of *exact* equations. The idea of *exactness* has its origin in the calculus of two (or more) variables. Recall that a differentiable function $u(x, y)$ of two variables has a differential

$$du = M(x, y) dx + N(x, y) dy \quad (1.33)$$

where $M = \partial u / \partial x$ and $N = \partial u / \partial y$. It follows from this that the functions M and N satisfy the relation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1.34)$$

throughout the domain D provided the function u has continuous second partial-derivatives there. Such a function is said to be of class C^2 in that

region, or, more briefly, is said to be C^2 there. The reason that equation (1.34) holds is that mixed second partial-derivatives of u are then equal: $\partial^2 u / \partial x \partial y = \partial u^2 / \partial y \partial x$. Moreover, the converse is also true if the domain D is simply connected (a domain is simply connected if it has no "holes." The interior of a circle or rectangle is simply connected, but that of an annulus is not). In other words, if equation (1.34) holds throughout such a region, then there exists a C^2 function u for which equation (1.33) holds. An equation

$$Mdx + Ndy = 0 \quad (1.35)$$

for which the condition (1.34) holds is called an *exact* equation. We suppose in the following discussion that the functions under consideration are sufficiently differentiable in a simply connected domain D .

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad (1.36)$$

this is the general differential equation in standard form, although now we have written x for the independent variable and y for the dependent variable. Rewrite equation (1.36) in the form

$$f(x, y) dx - dy = 0. \quad (1.37)$$

If there should be a function u such that

$$du = f(x, y) dx - dy, \quad (1.38)$$

then equation (1.37) requires that u be constant. Solving the equation $u(x, y) = C$ for y as a function of x should then provide the solution of the differential equation. However, this result is of little value because the condition (1.34), which is necessary for u to exist, implies that f is independent of y in D , that is, that the equation is of the form $y' = f(x)$. This is indeed solvable as in the trivial case of equation (1.12) above, but this is not interesting.

Observe, however, that the differential equation, in either of the forms (1.36) or (1.37), is unaffected by multiplying each side of the equation with a nonzero function $p(x, y)$. Doing so transforms equation (1.37) to (1.35) with $M = pf$ and $N = -p$. The condition (1.34) for the existence of a function u satisfying equation (1.33) is then

$$f \frac{\partial p}{\partial y} + \frac{\partial p}{\partial x} + p \frac{\partial f}{\partial y} = 0. \quad (1.39)$$

If such a function p can be found, the solution can be found by then finding the function u and setting $u(x, y) = C$ for an appropriate constant C . The following example shows that the method of solving the linear, inhomogeneous equation (1.20) is an example of the present method.

Example 1.3.5 Suppose the equation (1.36) is linear: $f(x, y) = a(x)y + b(x)$. Then equation (1.39) takes the special form

$$(a(x)y + b(x)) \frac{\partial p}{\partial y} + \frac{\partial p}{\partial x} + a(x)p = 0.$$

Observe that we can seek a solution of this partial differential equation in the form $p = p(x)$, independent of y , since that eliminates the first term and reduces the equation to the differential equation $dp/dx + a(x)p = 0$. This we can indeed solve, as in §1.2. It gives us precisely the factor K given in equation (1.21), if the changes of notation are noted. \square .

The preceding example is not complete: whereas an integrating factor has been found, the solution to the original problem requires a further step. That is to find the function $u(x, y)$ such that the solution to the original equation is given implicitly in the form $u(x, y) = C$. For this one uses equation (1.33), obtaining the function u via a line integral

$$u(x, y) = \int_{x_0, y_0}^{x, y} M(\xi, \eta) d\xi + N(\xi, \eta) d\eta$$

from some chosen point (x_0, y_0) to the current point (x, y) . The exactness of the form $Mdx + Ndy$, together with the fact that the region is simply connected, imply that the line integral is independent of the path and therefore that u is a function. Details are left to the reader.

The equation (1.39) is a partial-differential equation. Since partial-differential equations are typically more difficult to solve than are ordinary-differential equations, it may appear that very little progress has been made. There are, however, cases – usually involving educated guesses – for which one can indeed solve equation (1.39) explicitly for p , then find the function u , and finally solve the differential equation (1.36) by the method outlined in this section.

More generally, if the form $M dx + N dy$ is not exact but $p(M dx + N dy)$ is, we call p an integrating factor. It is a solution of the partial-differential equation

$$\frac{\partial(pM)}{\partial y} = \frac{\partial(pN)}{\partial x}.$$

1.3.3 Bernoulli and Riccati Equations

Some examples of nonlinear, first-order differential equations of special interest are those associated with the names Bernoulli and Riccati. Bernoulli's equation may be written

$$y' + a(x)y = b(x)y^n, \quad (1.40)$$

where a and b are given functions. The substitution $u = y^{1-n}$ converts this to the linear, inhomogeneous equation

$$u' + (1-n)a(x)u = (1-n)b(x),$$

which can be solved 'by quadratures' with the aid of equation (1.23) above.

A Riccati equation is one of the form

$$y' + a(x)y + b(x)y^2 = c(x). \quad (1.41)$$

It cannot in general be reduced to quadratures like the Bernoulli equation, but has the following special property. Suppose a solution $y = Y(x)$ is known. Then any *other* solution may be found by quadratures. To see this, let $y = Y + \eta$; the equation for η is

$$\eta' + (a(x) + 2Y(x))\eta = -b(x)\eta^2.$$

This is a Bernoulli equation with $n = 2$ so the substitution $\eta = y^{-1}$ reduces it to a linear equation which can be solved by quadratures.

There are other "named" equations, associated with the researches of particular mathematicians, but we pass on now to more general issues.

PROBLEM SET 1.3.1

1. A variant of the simple model of population growth in Example 1.2.1 asserts that birth rates get higher with overcrowding, so that the equation (1.18), where k is constant, should be replaced by the equation $\dot{x} = k_0x^{1+\epsilon}$, i.e., k should be replaced by k_0x^ϵ to allow for enhanced birthrates with increasing population. Here k_0 and ϵ are positive constants. Solve the initial-value problem for this equation explicitly and show that it predicts infinite population in finite time.

2. Assume that the functions g and f appearing in the formula (1.29) are continuous in open intervals containing the points y_0 and x_0 , respectively. Prove that this equation can be solved for y as a function x in a neighborhood of the point x_0 , provided only that $g(y_0) \neq 0$.

Hint: apply the Implicit-Function theorem to the equation $G(x, y) = 0$ where

$$G(x, y) = \int_{y_0}^y \frac{du}{g(u)} - \int_{x_0}^x f(v) dv.$$

3. Let ϕ be an arbitrary function defined and continuously differentiable (C^1) on an interval I of the x axis, and suppose it is not constant there. Show that, at least on some subinterval I' of I , there is a continuous function f such that the differential equation $y' = f(y)$ possesses the solution $y = \phi(x)$ on I' .
4. Find the general solution of the equation $xy' = y$ both by the formula (1.17) and by the substitution $y = xu$.

In the following two problems, reduce the equations to forms like that of equation (1.29) expressing the solutions implicitly but leave the expressions in the form of integrals.

5. $y' = (ax + by) / (cx + dy)$, a, b, c, d constants such that $ad \neq bc$ (why this restriction?).
6. $y' = \sin(y/x)$.
7. Consider the logistic equation (1.30) with initial value $x_0 > 0$ at $t = 0$. Express the solution explicitly in terms of elementary functions.
- What are its equilibrium values? (See Problem 10 of Problem Set 1.2.1 above for a definition)
 - Suppose $k > 0$. Find $\lim_{t \rightarrow +\infty} x(t)$. Does it depend on x_0 ?
 - Suppose $k < 0$. Can you find $\lim_{t \rightarrow +\infty} x(t)$? Does it depend on x_0 ?
8. Find the explicit solution of the equation

$$\dot{u} = u^2 - c^2$$

and express its dependence on the initial value u_0 , the initial time t_0 and the nonzero constant c . You may assume $u_0 \neq \pm c$.

9. Consider Example 1.3.3. Find an equilibrium solution $v = v_*$ (cf. Problem 10 of Problem Set 1.2.1). Solve equation 1.31 with the initial data $v(0) = 0$ and show that the solution approaches v_* as $t \rightarrow \infty$ (ignore the fact that the object will hit the ground in finite time). The velocity v_* is called the terminal velocity.
10. Suppose that $p(x, y)$ and $q(x, y)$ are both integrating factors for the form $M(x, y) dx + N(x, y) dy$. Show that $\alpha p + \beta q$ is also an integrating factor, for arbitrary constants α and β .
11. Refer to Example 1.3.5. Complete the solution of the general, linear differential equation by the method of exact equations along the line suggested immediately following that example.
12. Consider the separable equation (1.28), write it in the form (1.37) and find an integrating factor making it exact.
13. Show that $p(x, y) = 1/(ax^2 + 2bxy + cy^2)$ is an integrating factor for $ydx - xdy$, for any values of the constants a, b, c (not all zero).
14. Let $M(x, y) = yf(xy)$ and $N(x, y) = xg(xy)$, where $f(v)$ and $g(v)$ are functions of a single real variable v , defined and continuously differentiable for all real values of v . Under what conditions on f and g is the form $Mdx + Ndy$ exact for all values of x, y in the plane? In that case, find the function $u(x, y)$ such that equation (1.33) holds, and use that information to infer the general solution to the equation $y' = -(M(x, y)/N(x, y))$.
15. For the Riccati equation (1.41) show that the Bernoulli substitution $u = y^{-1}$ merely gives another Riccati equation. However, if a solution $y = Y(x)$ can be found, show that the substitution $y = Y + 1/u$ leads to a linear, inhomogeneous equation in u .
16. Use the technique of the preceding problem to reduce the Riccati initial-value problem

$$y' - y + 2x^{-3}y^2 = x^2, \quad y(1) = 0$$

to a specific linear, inhomogeneous problem. Hint: $Y(x) = -x^2$.

17. Consider the linear, second-order, differential equation

$$u'' + a(x)u' + b(x)u = 0,$$

and put

$$u = \exp \left\{ \int^x y(s) ds \right\}.$$

Show that y satisfies a Riccati equation.

18. Consider the specific, linear, second-order, differential equation

$$u'' - (1 + x^2)u = 0. \quad (1.42)$$

Verify that the substitution of the preceding problem leads to the Riccati equation

$$y' + y^2 = 1 + x^2,$$

for which $y = x$ is a solution. Use the technique of problem 15 to obtain a linear, inhomogeneous equation. Explain how this leads to a solution by quadrature of equation (1.42) (you do not need to carry this out).

1.4 Inequalities, Uniqueness and Existence

In general the initial-value problem,

$$\frac{dy}{dx} = f(x, y), \quad y|_{x=x_0} = y_0, \quad (1.43)$$

can neither be solved explicitly in terms of known elementary functions nor reduced to quadratures. The question arises: can it be solved at all? In other words, are there well-defined functions f for which there is no solution to the initial-value problem (1.43)? In fact, this problem possesses a unique solution under quite general, rather mild conditions on the function $f(x, y)$ appearing on the right-hand side. The proof of the theorem asserting the existence of a solution is deferred to Chapter 6), but it is stated in this section (Theorem 1.4.3).

By *uniqueness* is meant that only one solution can satisfy the initial condition $y(x_0) = y_0$ or, put another way, two distinct solutions cannot intersect at the point (x_0, y_0) . The uniqueness of the solution to the basic initial-value problem (1.13) can be inferred in a more elementary manner than its existence, and it is useful to address this now. It is based on a simple inequality, and we next turn to this.

1.4.1 Gronwall's Lemma

There is a very simple result, based on the same reasoning that gave us the formula (1.21) for the solution of the general linear problem, which is

“unreasonably” effective in the analysis of differential equations. It is the following:

Lemma 1.4.1 (*Gronwall’s Lemma*) *Let u be a continuous function on the interval $a \leq x \leq b$ and let α and β be constants with $\beta \geq 0$. If*

$$u(x) \leq \alpha + \beta \int_a^x u(s) ds \quad (1.44)$$

on this interval, then

$$u(x) \leq \alpha \exp(\beta(x-a)) \quad (1.45)$$

there.

Proof: Define

$$R(x) = \int_a^x u(s) ds.$$

Then

$$\frac{dR}{dx} = u(x) \leq \alpha + \beta R(x),$$

so, multiplying through by $\exp(-\beta x)$, we obtain

$$\frac{d}{dx}(R \exp(-\beta x)) \leq \alpha \exp(-\beta x).$$

Next, integrating from a to $x > a$ and noting that $R(a) = 0$, we find

$$\exp(-\beta x) R(x) \leq -\frac{\alpha}{\beta} (\exp(-\beta x) - \exp(-\beta a)),$$

so

$$R(x) \leq \frac{\alpha}{\beta} (\exp \beta(x-a) - 1).$$

Substituting this expression for the integral in equation (1.44) then gives the conclusion (1.45). \square

This version of Gronwall’s lemma holds when the current point x is greater than a . There is a corresponding version when $x < a$:

Corollary 1.4.1 *Suppose, under the same conditions on u, α, β that $c < x < a$ and*

$$u(x) \leq \alpha + \beta \int_x^a u(s) ds. \quad (1.46)$$

Then

$$u(x) \leq \alpha \exp(\beta(a-x)) \quad (1.47)$$

there.

The proof may be obtained by reducing this to the form given in the Lemma by means of the substitution $x \rightarrow 2a - x$.

1.4.2 The Lipschitz Condition

The differential equation is determined by the properties of the function on the right-hand side of equation (1.36), so we need to know something about this function before we can say anything about the solution. Most functions $f(x, y)$ encountered in practice are at least continuous, and usually have continuous derivatives (often infinitely many). A convenient assumption, which is general enough for most situations, is that f satisfy a *Lipschitz condition* in some region of the xy -plane. We define this first in the case of a function $f(y)$ that does not depend on the independent variable x :

Definition 1.4.1 *The function f defined for y in the interval I satisfies a Lipschitz condition there if there is a positive constant L such that*

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2| \quad (1.48)$$

for all y_1, y_2 in I .

It's easy to see from this definition that the function f is continuous in y but not necessarily differentiable. For example, the function $f(y) = |y|$ satisfies a Lipschitz condition everywhere but fails to be differentiable at $y = 0$.

In the general version of the initial-value problem, as in Theorem (1.4.3) below, the function f appearing on the right-hand side of the equation depends on x as well as y . The generalization of the Lipschitz condition that is needed for the uniqueness and existence theorems is placed on the y dependence only, and will be referred to as "a Lipschitz condition with respect to y ."

Definition 1.4.2 *The function f satisfies a Lipschitz condition with respect to y in a region D of the xy -plane if there is a positive constant L such that*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad (1.49)$$

for each pair $(x, y_1), (x, y_2)$ in D .

If we assume that f is a continuous function on D and satisfies the condition (1.49), we have then made an assumption stronger than continuity but weaker than differentiability.

There is a simple, sufficient condition for a function f to satisfy a Lipschitz condition with respect to y in a region D of the xy -plane. One of the

requirements in this condition is that the region be *convex*. A convex region C is one with the property that if a pair of points p_1 and p_2 belongs to C , so also does the entire line segment joining them. The region inside a circle or a rectangle is convex, whereas that inside an annulus is not.

Theorem 1.4.1 *Suppose f and its first partial derivatives are continuous in the closed, bounded and convex region D of the xy plane. Then f satisfies a Lipschitz condition with respect to y there.*

Proof: We recall from analysis the theorem that a continuous function on a closed, bounded set has a maximum and a minimum there. Consequently, there is some number L such that

$$\left| \frac{\partial f}{\partial y} \right| \leq L$$

on D . By the mean-value theorem (with fixed x),

$$f(x, y_1) - f(x, y_2) = \frac{\partial f}{\partial y}(x, y^*)(y_1 - y_2),$$

where y^* lies between y_1 and y_2 . Taking absolute values and applying the preceding inequality, we immediately obtain the required result. \square

The assumptions in this theorem are somewhat stronger than necessary for the result. For example, if the domain is unbounded but the partial derivative is nevertheless known to be bounded, the conclusion holds. Furthermore, the convexity condition is stronger than is needed: it would suffice if the vertical lines from (x, y_1) to (x, y_2) lie in D for each fixed x . This formulation is nevertheless convenient since it is easy to verify in a great many cases of interest.

We can now state and prove a uniqueness theorem for the initial-value problem (1.43).

Theorem 1.4.2 *Suppose $y = \varphi(x)$ is a solution of the initial-value problem (1.43) on an interval $c < x < b$, with graph $(x, \varphi(x))$ remaining in a region D of the xy -plane in which f is continuous and satisfies a Lipschitz condition with respect to y . Then φ is the only such solution.*

Proof: Since $\varphi(x)$ is a solution it satisfies the relation $\varphi'(x) = f(x, \varphi(x))$, and furthermore $\varphi(x_0) = y_0$. Integrating the equation for φ from x_0 to x and taking the initial condition into account gives the equation

$$\varphi(x) = y_0 + \int_{x_0}^x f(s, \varphi(s)) ds.$$

Suppose now there is a second solution ψ having the same properties, i.e., it has the same initial condition and its graph remains in D on the same interval (c, b) . It satisfies the same equation so, subtracting, we obtain

$$\varphi(x) - \psi(x) = \int_{x_0}^x \{f(s, \varphi(s)) - f(s, \psi(s))\} ds.$$

Taking absolute values gives

$$|\varphi(x) - \psi(x)| \leq \int_{x_0}^x |f(s, \varphi(s)) - f(s, \psi(s))| ds$$

where we have assumed for definiteness that $x > x_0$. The integrand can be estimated with the aid of the Lipschitz condition:

$$|f(s, \varphi(s)) - f(s, \psi(s))| \leq L |\varphi(s) - \psi(s)|.$$

Then, defining $u = |\varphi - \psi|$, we see from the preceding two inequalities that

$$u(x) \leq L \int_{x_0}^x u(s) ds$$

for $x_0 < x < b$. Referring to Gronwall's lemma with $\beta = L$ but $\alpha = 0$, we infer that $u \leq 0$ on this entire interval. But in view of its definition $u \geq 0$ at each point. Therefore $u \equiv 0$ on $[x_0, b]$, i.e., $\psi = \varphi$ there. A similar analysis on the interval $c < x < x_0$ using the corollary to Gronwall's lemma shows that this holds on that interval as well. \square

The Lipschitz condition with respect to y is sufficient for uniqueness but not necessary. For example, the following, somewhat less stringent, condition suffices as well. Consider the initial-value problem (1.43) and suppose that, in place of the inequality (1.49), we have the inequality

$$|f(x, y_1) - f(x, y_2)| \leq \phi(|y_1 - y_2|), \quad (1.50)$$

where ϕ is defined on an interval $[0, a]$ of the real axis ($a > 0$), is continuous and increasing there, and satisfies the two conditions

$$\phi(0) = 0 \text{ and } \int_x^a \frac{du}{\phi(u)} \rightarrow \infty \text{ as } x \rightarrow 0+. \quad (1.51)$$

For example, in the case of the Lipschitz condition, $\phi(u) = Lu$. The proof of uniqueness under the conditions (1.50) is considered in the next problem set.

1.4.3 The Existence Theorem

The uniqueness theorem 1.4.2 asserts that there cannot be more than one solution of the initial-value problem but does not assert that there is one. In fact, under the same conditions as those of the uniqueness theorem, there does indeed exist a solution.

Theorem 1.4.3 *Suppose the function f of equation (1.43) is continuous and satisfies a Lipschitz condition with respect to y in a domain D of the xy -plane containing the point (x_0, y_0) . Then there exists an interval $a < x < b$ containing x_0 on which a solution φ of (1.43) exists; the graph $(x, \varphi(x))$ lies in D on this interval.*

The proof of this theorem is deferred to Chapter 6.

The existence theorem is local: it asserts the existence of a solution curve only in a neighborhood of the point (x_0, y_0) where the initial data are assigned. Thus the interval (c, b) might be very small. The situation is actually better than it appears in that the interval of existence of the solution can often be extended in a very natural way. This too is taken up in Chapter 6.

We are not through with first-order equations. We'll need some other properties of them from time to time as we explore other subjects, but we'll develop these properties as we need them.

PROBLEM SET 1.4.1

1. There is a discrete version of Gronwall's lemma. Let $u_1, u_2, \dots, u_n, \dots$ be a sequence (finite or infinite) of numbers satisfying the inequalities

$$u_1 \leq \alpha \text{ and } u_{n+1} \leq \alpha + \beta \sum_{j=1}^n u_j, \quad n = 1, 2, \dots$$

Assume $\beta > 0$ and derive the inequality

$$u_n \leq \alpha (1 + \beta)^{n-1}.$$

2. Suppose the inequality (1.44) is replaced by

$$u(x) \leq \alpha(x) + \int_a^x \beta(s) u(s) ds,$$

where α and β are continuous functions on $[a, b]$, β nonnegative and α non-increasing there. Find the generalization of Gronwall's lemma to this case.

3. Find a Lipschitz constant L such that $|f(x, y) - f(x, z)| \leq L|y - z|$ for the given function f in the given domain:

- (a) $f(x, y) = x^2 + y^2$, $x \geq 0, |y| \leq 2$.
- (b) $f(x, y) = xy^2$, $0 \leq x \leq 2, |y| \leq 2$.
- (c) $f(x, y) = \cos(2y)$, $|y| \leq \pi/6$.
- (d) $f(x, y) = x/(1 + y^2)$, $|x| \leq 1, |y| \leq 1$.

4. Consider the initial-value problem

$$\dot{x} = \sqrt{x}, \quad x(0) = 0.$$

- Find two solutions on the interval $[0, 1]$.
- Show that the function $f(x) = \sqrt{x}$ cannot satisfy a Lipschitz condition on an interval of the form $[0, a]$ with $a > 0$.

5. Consider the initial-value problem

$$\dot{x} = 3x^{2/3}, \quad x(0) = 0.$$

- Show that the family of functions (for $c > 0$)

$$x_c(t) = \begin{cases} 0 & \text{if } -\infty < t \leq c \\ (t - c)^3 & \text{if } c < t < \infty \end{cases}$$

represent a one-parameter family of solutions of the initial-value problem; graph a couple of them.

- Show that the function $f(x) = x^{2/3}$ cannot satisfy a Lipschitz condition on an interval of the form $[0, a]$ with $a > 0$.

6. Suppose that the function $f(x, y)$ satisfies the condition for existence and uniqueness of solutions as in Theorem 1.4.3 above. Let the points (x_0, y_0) and $(x_0, y_0 + \eta)$ both lie in the domain described there, and suppose that the solution $y(x)$ with initial value y_0 at $x = x_0$ and the solution $y_\eta(x)$ with initial value $y_0 + \eta$ both exist on the interval (a, b) . Show that

$$|y(x) - y_\eta(x)| \leq |\eta| e^{L|x-x_0|}$$

on (a, b) . Explain why this implies that solutions are continuous functions of their initial values.

7. Where is the assumption of convexity used in the proof of Theorem 1.4.1?
 8. Consider the equation

$$\frac{dy}{dx} = \sqrt{x^2 - y^2}.$$

Describe the domain where this equation is defined, assuming only real values are allowed. Show that on the domain

$$D : |x| < K, \quad x^2 - y^2 > \Delta$$

where K and Δ are positive constants with $\Delta < K^2$, the function $\sqrt{x^2 - y^2}$ satisfies a Lipschitz condition, and estimate the Lipschitz constant L .

9. For the differential equation of the preceding problem with initial data $y = 1$ when $x = -2$, show that the interval of existence of the solution cannot extend to the origin, i.e., the solution cannot exist on the interval $(-2, 0)$.
 10. Prove the uniqueness of solutions when the conditions of equations (1.50) and (1.51) replace the Lipschitz condition.

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