Chapter 2

The General, Linear Equation

Let $a_1(x), a_2(x), \ldots, a_n(x)$ be continuous functions defined on the interval $a \leq x \leq b$, and suppose $u(x)$ is $n$-times differentiable there. We form from $u$ the function $Lu$, defined as follows:

$$(Lu)(x) = u^{(n)}(x) + a_1(x)u^{(n-1)}(x) + \cdots + a_{n-1}(x)u'(x) + a_n(x)u(x). \quad (2.1)$$

This expression for $L$ in terms of $u$ and its derivatives up to order $n$ is linear in $u$ in the sense that

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv \quad (2.2)$$

for any constants $\alpha, \beta$ and any sufficiently differentiable functions $u, v$. The corresponding linear differential equation has, as in the case of first-order equations, homogeneous and inhomogeneous versions:

$$Lu = 0 \quad (2.3)$$

and

$$Lu = r, \quad (2.4)$$

where, in the latter equation, the function $r$ is prescribed on the interval $[a, b]$. These equations are in standard form, that is, the coefficient of the highest derivative is one. They are sometimes displayed in an alternative form in which the highest derivative is multiplied by a continuous function $a_0(x)$. As long as this function does not vanish on $[a, b]$ the two forms are interchangeable, and we'll use whichever is the more convenient.
The linearity, as expressed by equation (2.2), plays a powerful role in the theory of these equations. The following result is an easy consequence of that linearity:

**Lemma 2.0.2** *Equation (2.2) has the following properties:*

- Suppose $u_1$ and $u_2$ are solutions of the homogeneous equation (2.3); then for arbitrary constants $c_1$ and $c_2$, $u = c_1 u_1 + c_2 u_2$ is also a solution of this equation.

- Suppose $u_1$ and $u_2$ are solutions of the inhomogeneous equation (2.4); then $u = u_1 - u_2$ is a solution of the homogeneous equation (2.3).

- Suppose $u_1$ and $u_2$ are solutions of the inhomogeneous equations $L u_1 = r_1$ and $L u_2 = r_2$; then for arbitrary constants $c_1$ and $c_2$, $u = c_1 u_1 + c_2 u_2$ is a solution of the inhomogeneous equation $L u = r$ where $r = c_1 r_1 + c_2 r_2$.

Linear equations are the most widely studied of all classes of differential equations. This is in part because they arise in a variety of applications, in part because an understanding of them is needed in studying more general, nonlinear differential equations, and in part for the rather unheroic reason that they represent the broadest class of ordinary differential equations for which a comprehensive theory exists. Much of this book is occupied with their study.

Most of the features of linear differential equations of order $n$ are already evident when $n = 2$. Moreover, some of the more important applications involve second-order differential equations. We’ll therefore take these up first. This chapter is divided into three sections. The first and most extensive is devoted to second-order differential equations. The second extends these results to the $n$th-order case. The third provides the first introduction to a modest reformulation that will find repeated application in subsequent chapters, that of a system of $n$ first-order equations.

### 2.1 Second-order linear equations

The linear operator $L$ is defined by its action on twice-differentiable functions:

$$Lu = u'' + p(x)u' + q(x)u$$  \hspace{1cm} (2.5)

where $p$ and $q$ are continuous functions on an interval $[a,b]$. The solution of the differential equation (either the homogeneous equation $Lu = 0$ or
the inhomogeneous equation $Lu = r$ with $r \neq 0$) involves two arbitrary constants. This can be seen in the simple example $u'' = 1$, which has the general solution $u = c_0 + c_1 x + (1/2) x^2$, for any values of the constants $c_0$ and $c_1$. These constants can be determined if we specify not only the function $u$ but also the first derivative $u'$ at a chosen point $x_0$. This should be contrasted with the situation in the preceding chapter on first-order equations, where specification of the function alone at a chosen point was sufficient to make the solution unique. The difference is the order of the equation and we shall find (see §2.2) that for an equation of order $n$, specification of the function and its first $n - 1$ derivatives at a chosen point are needed for uniqueness of the solution.

The initial-value problem now takes the form

$$Lu = r \text{ for } a \leq x \leq b, \quad u(x_0) = c_1, \quad u'(x_0) = c_2,$$

where $r$ is a function given on the interval $I = [a, b]$. That is, the differential equation is supposed to be satisfied on the interval and, at some point $x_0 \in I$, the solution $u$ and its first derivative are supposed to take on prescribed values. We again borrow an existence result (Theorem 6.2.3) from Chapter 6:

**Theorem 2.1.1** Suppose the functions $p, q, r$ are continuous on $[a, b]$. Then the initial-value problem (2.6) has a solution $u(x)$ on that interval.

Remarks:

- For $u$ to be a solution, it is understood that it must have derivatives up to order two at each point. The equation itself then requires that the second derivative be continuous.

- The existence theorem for the general (i.e., nonlinear) initial-value problem is local, i.e., guarantees a solution only on some interval, possibly small, containing the initial value $x_0$ (cf. Theorem 1.4.3 above). By contrast, the theorem above is global, i.e., guarantees a solution on the full interval $[a, b]$. The difference is in the present assumption of linearity.

- We have indicated a closed interval $[a, b]$, i.e., one that includes its endpoints. This choice is made for definiteness only and is in no way essential, but we shall indicate explicitly any deviation from this choice. At the endpoints of such an interval the derivatives are interpreted as one-sided.
The interval may be infinite or semi-infinite without changing the conclusion. If the coefficient functions $p, q, r$ are continuous on (say) all of $R$, then they are continuous on arbitrary finite intervals $[a, b]$ so solutions exist on these arbitrary intervals and therefore on all of $R$.

The uniqueness of this solution can be inferred from a simple and striking result about the homogeneous problem (2.3), which here takes the form

$$u'' + p(x) u' + q(x) u = 0, \ a \leq x \leq b. \quad (2.7)$$

**Lemma 2.1.1** Suppose $u$ is a solution of equation (2.7) satisfying “zero” initial data at some point $x_0 \in [a, b]$:

$$u(x_0) = 0, \ and \ u'(x_0) = 0.$$  

Then $u(x) = 0$ at every point $x \in [a, b]$.

Proof: Define

$$\sigma(x) = u(x)^2 + u'(x)^2.$$  

Note that $\sigma(x) \geq 0$ and that $\sigma(x_0) = 0$. Differentiating we find

$$\sigma' = 2(uu' + u'u'').$$

Substitute for $u''$ from the differential equation:

$$\sigma' = 2 \left((1 - q) uu' - pu'^2\right) \leq 2 \left((1 + |q|)|uu'| + |p|u'^2\right),$$  

where we have, in the last form of this inequality, noted that any real number is less than or equal to its absolute value. The continuity of the functions $q$ and $p$ means each is bounded by some constant on $[a, b]$: $|q| \leq Q, \ |p| \leq P$, where $Q$ and $P$ are constants. Therefore

$$\sigma' \leq 2 \left((1 + Q)|uu'| + Pu'^2\right).$$

Now from the identity $a^2 + b^2 \geq 2ab$ we infer that $2|uu'| \leq \left(u^2 + u'^2\right)$, and therefore that

$$\sigma' \leq K \left(u^2 + u'^2\right) = K \sigma \quad (2.8)$$

for the constant $K = 1 + Q + 2P$. Integrating from $x_0$ to a greater value $x$, we find (mindful that $\sigma(x_0) = 0$)

$$\sigma(x) \leq K \int_{x_0}^{x} \sigma(s) \ ds \ for \ x \geq x_0.$$
Gronwall’s lemma 1.4.1 now shows that $\sigma \equiv 0$ for $x \geq x_0$. Reasoning similar to that which led to the inequality (2.8) also shows that $\sigma' \geq -K\sigma$. Integrating $\sigma' \geq -K\sigma$ from $x$ to $x_0$ if $x \leq x_0$ and again applying Gronwall’s lemma, we find that $\sigma \equiv 0$ also for $x < x_0$, and therefore $\sigma$ vanishes at every point of the interval. In virtue of its definition, this proves the lemma. □

It is now easy to see that the initial-value problem (2.6) has at most one solution. For suppose $u$ and $v$ are both solutions. Then if $w = u - v$, $w$ is a solution of the homogeneous problem (2.7) (cf. Lemma 2.0.2). Furthermore, $w$ and $w'$ vanish at $x_0$. It follows from Lemma 2.1.1 that $w \equiv 0$ on $[a,b]$, i.e., that $u \equiv v$ on $[a,b]$. We summarize this as follows.

**Theorem 2.1.2** Under the assumptions of Theorem 2.1.1, there is a unique solution of the initial-value problem (2.6).

A zero of a function $u$ is a value of the independent variable $x$ such that $u(x) = 0$. A simple zero is one at which $u$ vanishes but $u'$ does not; a double zero $x_0$ is one at which $u(x_0) = u'(x_0) = 0$ but $u''(x_0) \neq 0$; and so on. Lemma 2.1.1 has certain consequences for the nature of the zeros of solutions of the homogeneous equation (2.7). For example, if $u$ is such a solution, it cannot have a double zero at any point of the interval unless it is the solution $u \equiv 0$. In general, a function may have infinitely many distinct zeros on a given interval $[a,b]$. However, if the function in question is the solution of the homogeneous differential equation (2.7), it can have only finitely many (except in the trivial case when it is identically zero). This is a consequence of Lemma 2.1.1 together with the following standard theorem of real analysis (see, for example, [12]):

**Theorem 2.1.3** From any infinite set of points lying in a closed and bounded subset $K$ of $R^k$ one can select an infinite sequence $\{x_n\}$ of distinct points converging to a point $x_*$ in $K$.

The set $[a,b]$ in $R$ is closed and bounded, so this theorem applies. Let $u$ be a solution of equation (2.7) other than the identically-zero solution, and suppose that $u$ has infinitely many zeros in $[a,b]$; these are necessarily simple by the preceding paragraph. By the Theorem 2.1.3 one can find a sequence $\{x_n\}$ of distinct zeros converging to $x_* \in [a,b]$. Since $u(x_n) = 0$, it follows from the continuity of $u$ that $u(x_*) = 0$. If $u'(x_*)$ vanished as well, we would have to conclude that $u$ is identically zero on $[a,b]$. This is contrary to our assumption, so $u'(x_*) \neq 0$. It follows that there is a neighborhood $N$ of $x_*$ in $[a,b]$ such that $u \neq 0$ in $N$, except at $x_*$ itself. But any neighborhood of $x_*$ contains infinitely many zeros of $u$. This contradiction proves the assertion, which we summarize as follows:
Theorem 2.1.4 Let the hypotheses of Theorem 2.1.1 hold, with \( r \equiv 0 \). Then a nontrivial solution \( u \) has at most finitely many zeros on \([a, b]\).

2.1.1 The homogenous problem and bases

Lemma 2.1.1 has the remarkable consequence that one can define a pair of solutions of the homogenous equation \( u'' + pu' + qu = 0 \) (eq. 2.7 above), say \( u_1(x) \) and \( u_2(x) \), with the property that any other solution \( u \) of that equation may be written as a linear combination of \( u_1 \) and \( u_2 \) on \([a, b]\):

\[
u(x) = c_1 u_1(x) + c_2 u_2(x), \tag{2.9}
\]

where \( c_1 \) and \( c_2 \) are constants.

To see this, define a pair of solutions \( u_1, u_2 \) of the initial-value problem as follows: \( u_1 \) is a solution of the homogenous equation (2.7) which also satisfies the initial data

\[
u_1(a) = 1, \quad u'_1(a) = 0, \tag{2.10}
\]

whereas \( u_2 \) is a solution of the homogenous equation (2.7) which also satisfies the initial data

\[
u_2(a) = 0, \quad u'_2(a) = 1. \tag{2.11}
\]

Each of these functions is uniquely determined, in view of the existence and uniqueness theorems 2.1.1 and 2.1.2. Now let \( v(x) \) be any solution whatever of the homogenous equation (2.7). Define \( c_1 = u(a) \) and \( c_2 = u'(a) \), and define a function \( v \) by the formula

\[
v(x) = c_1 u_1(x) + c_2 u_2(x).
\]

Then \( v \) is a solution of the homogenous equation which, because of equations (2.10) and (2.11), has the same initial data as does \( u \). Therefore \( v \equiv u \). This proves that equation (2.9) holds.

Example 2.1.1 Let the homogenous equation be \( u'' + u = 0 \), and take the left-hand endpoint to be \( a = 0 \). It is easy to verify that the functions \( \cos x \) and \( \sin x \) are solutions of the homogenous equation satisfying the initial data of equations (2.10) and (2.11), respectively. Therefore, any solution \( u(x) \) of the homogenous equation can be expressed in the form \( u(x) = c_1 \cos x + c_2 \sin x \) for some choice of the constants \( c_1 \) and \( c_2 \). □

The functions \( u_1 \) and \( u_2 \), defined above as solutions of the homogenous equation (2.7) together with the initial data (2.10) and (2.11), possess two important properties. One is their linear independence:
**Definition 2.1.1** A set of \( k \) functions \( u_1, u_2, \ldots, u_k \) defined on an interval \( I \) is linearly dependent there if there exist constants \( c_1, c_2, \ldots, c_k \), not all zero, such that

\[
c_1 u_1(x) + c_2 u_2(x) + \cdots + c_k u_k(x) = 0
\]

at each point of \( I \). A set of functions that is not linearly dependent on \( I \) is said to be linearly independent there.

This definition makes no reference to differential equations. Linear dependence or independence is a property that any collection of functions may or may not have.

**Example 2.1.2** Consider the functions \( \cos x \) and \( \sin x \) of the preceding example. For the sake of simplicity, assume that the interval is the entire \( x \) axis. If these functions are linearly dependent, then

\[
c_1 \cos x + c_2 \sin x = 0
\]

for all real values of \( x \). Setting \( x = 0 \) we infer that \( c_1 = 0 \), and then \( c_2 \) must also vanish since \( \sin x \) is not identically zero. \( \square \)

**Example 2.1.3** The functions \( 1, x, x^2, \ldots, x^k \) are linearly independent on any interval. To see this we write the expression for linear dependence:

\[
c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k = 0.
\]

Because this must hold on an interval, we can differentiate any number of times. If we differentiate \( k \) times we immediately infer that \( c_k = 0 \). Returning to the expression above, for which the highest degree term is now \( k - 1 \), we repeat the procedure and find that \( c_{k-1} = 0 \). Continuing, we find that all the constants must vanish, i.e., there can be no expression of linear dependence.

An alternative way of drawing this conclusion is proposed in Problem 7 in the problems at the end of this section. \( \square \)

The preceding examples are typical of proofs of linear independence: assume the opposite and deduce a contradiction in the form that all the coefficients must vanish. Let’s apply this approach to the functions \( u_1 \) and \( u_2 \) satisfying the conditions (2.10) and (2.11), respectively. Suppose that for some choice of constants \( c_1 \) and \( c_2 \)

\[
c_1 u_1(x) + c_2 u_2(x) = 0
\]

for all \( x \in [a, b] \). By virtue of condition (2.10) we find, by evaluating the left-hand side at \( x = a \), that \( c_1 = 0 \). Since the relation of linear dependence
above must hold for all \( x \in [a, b] \), we are free to differentiate it. Doing so and applying the condition (2.11), we then see that \( c_2 = 0 \). Thus these functions are linearly independent. Moreover, they are linearly independent by virtue of the conditions (2.10) and (2.11) without regard to whether they satisfy the differential equation or, indeed, any other requirement.

However, the second important property of the functions \( u_1 \) and \( u_2 \) is precisely that they are solutions of the linear, homogeneous differential equation (2.7). Such a linearly independent set of solutions is called a basis:

**Definition 2.1.2** A linearly independent pair of solutions of the second-order, linear, homogeneous equation (2.7) is called a basis of solutions for that equation.

We made a particular choice of basis \((u_1, u_2)\) above, but bases are not unique. The choice made for \((u_1, u_2)\), by requiring them to satisfy the conditions (2.10, 2.11), is useful for initial data that are supplied at the point \( x = a \), and is sometimes referred to as the standard basis for problems in which data are prescribed at \( x = a \). We could, however, have chosen some point other than \( x = a \) to supply the same initial values of \( u_i \) and \( u_i' \) \((i = 1, 2)\), and the result would have been a different but equally valid basis. Alternatively, we could have chosen different initial values for \( u_i \) and \( u_i' \) and the result would again be a basis, unless these values are chosen “peculiarly,” in a sense described below, following Theorem 2.1.7.

These two properties of the pair \((u_1, u_2)\) (that they are linearly independent and are solutions of the homogeneous equation (2.7)) imply that any solution of equation (2.7) is a linear combination of them with constant coefficients, as in equation (2.9). We now show that this property is common to all bases of solutions of the linear, homogeneous equation (2.7):

**Theorem 2.1.5** Let \( v_1 \) and \( v_2 \) be a basis of solutions for equation (2.7) and let \( u \) be any solution of that equation. Then there exist constants \( c_1 \) and \( c_2 \) such that

\[
 u(x) = c_1 v_1(x) + c_2 v_2(x) \quad \forall x \in [a, b]. \tag{2.12}
\]

**Proof:** Pick any point \( x_0 \) in \([a, b]\) and consider the pair of equations

\[
\begin{align*}
 c_1 v_1(x_0) + c_2 v_2(x_0) &= u(x_0) \\
 c_1 v_1'(x_0) + c_2 v_2'(x_0) &= u'(x_0).
\end{align*}
\tag{2.13}
\]

This is a system of two linear equations in the two unknowns \( c_1 \) and \( c_2 \). For such a system there are two possibilities\(^1\): either (1) the system (2.13) has

\(^1\)We rely here on a basic theorem in linear algebra; see, for example, [9].
a unique solution \((c_1, c_2)\) or (2) the corresponding homogeneous system

\[
\begin{align*}
c_1 v_1 (x_0) + c_2 v_2 (x_0) &= 0 \\
c_1 v_1' (x_0) + c_2 v_2' (x_0) &= 0
\end{align*}
\]  

(2.14)

has a nontrivial solution, i.e., a solution for which \(c_1\) and \(c_2\) are not both zero. Suppose that it is this second alternative that prevails. Then the function

\[
v (x) = c_1 v_1 (x) + c_2 v_2 (x)
\]

is a solution of equation (2.7) for which \(v (x_0) = 0\) and \(v' (x_0) = 0\). By Lemma 2.1.1 the function \(v\) vanishes identically on the interval \([a, b]\). But this implies that \(v_1\) and \(v_2\) are linearly dependent, contrary to the hypothesis that they are a basis. Consequently the second alternative is not possible, and the first must hold. But then the functions \(u\) and \(c_1 v_1 + c_2 v_2\) are both solutions of the same initial-value problem, and are therefore the same by Theorem 2.1.2. \(\square\)

Some simple cases, for which solutions are elementary functions, follow.

**Example 2.1.4** Return to the equation \(u'' + u = 0\) of Example 2.1.1 above; it has the solutions \(u_1 = \cos x\) and \(u_2 = \sin x\), which are linearly independent, as shown in Example 2.1.2. These constitute a basis, the standard basis relative to the point \(x = 0\). However, consider the pair \(v_1 = \cos x - \sin x\) and \(v_2 = \cos x + \sin x\). Since \(u_1 = (v_1 + v_2)/2\) and \(u_2 = (v_1 - v_2)/2\), the relation (2.9) shows that the arbitrary solution \(u\) can be written as a linear combination of \(v_1\) and \(v_2\). This pair therefore constitutes a second basis.

**Example 2.1.5** It is easy to verify that the equation

\[
u'' - u = 0
\]

(2.15)

has the solutions \(u_1 = \exp x\) and \(u_2 = \exp (-x)\); these are linearly independent (cf. Problem 12 in the problem set for this section). On the other hand, recall that \(\cosh x = (u_1 + u_2)/2\) and \(\sinh x = (u_1 - u_2)/2\). Thus \(v_1 = \cosh x\) and \(v_2 = \sinh x\) constitute a second basis (it is this latter basis which represents a standard basis relative to the origin for this equation.)

A linear vector space \(V\) over the real numbers is defined by a set of axioms\(^2\). If the space has a finite basis, i.e., a set of (say) \(n\) linearly independent elements (vectors) \(v_1, v_2, \ldots, v_n\) such that any vector \(v\) in \(V\) may be written

\[
v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n
\]

\(^2\)See, for example [9]
for some choice of the real numbers \(c_1, c_2, \ldots, c_n\), then the space is said to be finite-dimensional (\(n\)-dimensional in this case). Otherwise it is infinite-dimensional.

The set \(C^2[a,b]\) of real-valued functions possessing continuous second derivatives on the interval \([a,b]\) satisfies the axioms for a vector space. It is easily seen to be infinite-dimensional. However the subset of \(C^2[a,b]\) consisting of those functions satisfying a linear, homogeneous differential equation like equation (2.7) is also a vector space, a subspace of \(C^2[a,b]\). This vector space has a basis consisting of two elements. It follows that the solution space of equation (2.7) is a two-dimensional, real vector space.

### 2.1.2 The Wronskian

The Wronskian determinant of a pair of functions is defined as follows:

**Definition 2.1.3**

\[
W(u, v; x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v. \quad (2.16)
\]

We often write \(W(u, v; x) = W(x)\) for short. When we do this we need to remember that we have a specific pair of functions \(u\) and \(v\) in mind, and that the sign of the Wronskian changes if we interchange them.

**Theorem 2.1.6** Let \(u\) and \(v\) be solutions of equation (2.7). If \(x_0\) is any point of \([a,b]\) and \(W_0\) the value of the Wronskian there, then, at any point \(x\) of \([a,b]\),

\[
W(x) = W_0 \exp \left( -\int_{x_0}^x p(s) \, ds \right). \quad (2.17)
\]

Proof: Differentiating \(W\) as defined in equation (2.16) gives \(W' = uv'' - u''v\).

Substituting for the second derivatives from the equation (2.7), satisfied by each of \(u\) and \(v\), then gives \(W' = -pW\). The conclusion follows from this.

An immediate deduction from this is the following:

**Corollary 2.1.1** The Wronskian of a pair of solutions of a linear, homogeneous equation (2.3) either vanishes identically on the entire interval \([a,b]\) or does not vanish at any point of the interval.

The Wronskian is precisely the determinant appearing in the systems (2.13) and (2.14) above. When the Wronskian vanishes, the solutions \(v_1\)
and $v_2$ are linearly dependent, and when the Wronskian does not vanish they are linearly independent and therefore constitute a basis. This proves the following:

**Theorem 2.1.7** A necessary and sufficient condition for a pair $(u_1, u_2)$ of solutions of equation (2.7) to be a basis for that equation is that its Wronskian $W(u_1, u_2; x)$ not vanish on $[a, b]$.

Remark: If a pair of solutions $u_1$ and $u_2$ is chosen “at random,” i.e., if their initial data are assigned randomly, we would expect that $W \neq 0$; in this sense it would take a “peculiar” choice of solutions for that pair to be linearly dependent.

The expression (2.17) for the Wronskian may be used to find a formula for a second solution $v$ of equation (2.7) if we know one solution $u$. Suppose one such solution $u$ (not identically zero) is known on $[a, b]$. Pick a point $x_0$ where $u \neq 0$ and define

$$w(x) = c \exp \left\{ - \int_{x_0}^{x} p(s) \, ds \right\}$$

where $c$ is not zero but is otherwise arbitrary. Write

$$w(x) = u(x) v'(x) - u'(x) v(x) = u^2 \frac{d}{dx} \left( \frac{v}{u} \right),$$

so that

$$v(x) = u(x) \int_{x_0}^{x} \frac{w(s)}{u(s)^2} \, ds. \quad (2.18)$$

This formula defines $v$ on an interval containing $x_0$ on which $u$ does not vanish. It is straightforward to check that $v$ is indeed a solution of equation (2.7) on such an interval, and that the pair $u, v$ has the nonvanishing Wronskian $w$ there.

**PROBLEM SET 2.1.1**

1. Verify equation (2.2) in detail for the special case when $n = 2$ and the operator $L$ is defined by equation (2.5). Using equation (2.2), verify Lemma 2.0.2.

2. Let $L$ be the second-order operator defined by equation (2.5). Suppose $k$ functions $r_i(x), i = 1, \ldots, k$ are given on the interval $[a, b]$ and the corresponding $k$ initial-value problems

$$Lu_i = r_i, \quad u_i(x_0) = u'_i(x_0) = 0, \quad (i = 1, \ldots, k) \quad (2.19)$$
are solved for the $k$ functions $u_i$, $i = 1, \ldots, k$. Define $r(x) = \sum_{i=1}^{k} r_i(x)$.

Show that the unique solution of the initial value problem $Lu = r$, $u(x_0) = u'(x_0) = 0$ is

$$u(x) = \sum_{i=1}^{k} u_i(x).$$

3. In equation (2.7) let the interval be $[-1, 1]$. Can continuous coefficients $q$ and $p$ be chosen such that $u(x) = x^2$ is a solution of this equation? Explain.

4. Let $u(x) = x^3$ on $[-1, 1]$ and define $v(x) = -x^3$ on $[-1, 0]$ and $v(x) = +x^3$ on $(0, 1]$. Verify that $v$ is $C^2$ on $[-1, 1]$. Calculate $W(u, v; x)$. Show that these functions linearly independent on $[-1, 1]$?

5. The preceding problem shows that the vanishing of the Wronskian need not imply linear dependence in general, i.e., for functions not obtained as solutions of a homogeneous, linear differential equation. However, show that if functions $u_1(z), u_2(z)$ are analytic functions in a domain $D$ of the complex plane whose Wronskian determinant vanishes at each point of $D$, they must indeed be linearly dependent there over the complex numbers.

6. Extend the result of the preceding problem to $n$ complex functions: if $\{u_j(z)\}_{j=1}^{n}$ is a set of $n$ complex functions whose Wronskian is defined and vanishes on a domain $D$ of the complex plane, then they are linearly dependent there over the complex numbers.

7. The fundamental theorem of algebra states that a polynomial of degree $n$ can have no more than $n$ distinct zeros. Draw the conclusion of Example 2.1.3 from this.

8. Show that the three functions $\sin(x), \sin(2x), \sin(3x)$ are linearly independent on any nontrivial interval of the $x$ axis.

9. For the equation $u'' + u = 0$, $\cos(x)$ and $\sin(x)$ form a basis of solutions. Verify that $\cos(x + a)$ and $\sin(x + a)$ are also solutions if $a$ is any constant. Use this and the uniqueness theorem to infer the trigonometric addition theorems

$$\cos(x + a) = \cos a \cos x - \sin a \sin x,$$
$$\sin(x + a) = \cos a \sin x + \sin a \cos x.$$

10. Find bases of solutions for the following equations:

     a) $u'' = 0$; b) $u'' + 2u' = 0$; c) $u'' + xu' = 0$.

11. Consider the equation $u'' + u = 0$. Under what conditions on the constants $a, b, c, d$ is the pair

     $v_1 = a \cos x + b \sin x, \ v_2 = c \cos x + d \sin x$

     a basis of solutions for this equation?
12. Consider the equation \( u'' - u = 0 \). Under what conditions on the constants \( a, b, c, d \) is the pair
\[
v_1 = ae^x + be^{-x}, \quad v_2 = ce^x + de^{-x}
\]
a basis of solutions for this equation?

13. Show that the functions \( u_1(x) = 1/x \) and \( u_2(x) = 1/x^2 \) form a basis of solutions for the equation
\[
u'' + \frac{4}{x}u' + \frac{2}{x^2}u = 0
\]
on any interval excluding the origin.

14. Same as the preceding problem for the functions \( u_1 = \sin x - x \cos x \), \( u_2 = \cos x + x \sin x \) and the differential equation
\[
u'' - \frac{2}{x}u' + u = 0.
\]

15. Verify that the second solution given in the formula (2.18) is linearly independent of \( u \) on intervals on which \( u \) does not vanish.

16. Let \( u \) be a solution of (2.7) vanishing at the endpoints \( x_1 \) and \( x_2 \) of an interval but not vanishing in between. Show that the formula (2.18), with \( x_1 < a < x_2 \), leads to finite limiting values for the second solution \( v \) at \( x_1 \) and \( x_2 \), and find these limits.

17. A second-order equation
\[
\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u = 0
\]
is said to be in "self-adjoint" form. What condition on the function \( p \) is needed for the existence theorem 2.1.1 to hold? Find an explicit expression for the Wronskian in terms of the function \( p \).

18. For the equation
\[
\frac{d}{dx} \left( (1 - x^2) \frac{du}{dx} \right) + 2u = 0
\]
• On what intervals does the existence theorem guarantee a solution?
• Verify that \( u_1 = x \) is a solution.
• Find a second solution in an interval containing the origin. How far can this interval extend on each side of the origin?

19. Let \( u \) and \( v \) be \( C^2 \) functions on an interval \( I \), and suppose their Wronskian \( uv' - u'v \) does not vanish at any point of \( I \). Show that their zeros separate each other, i.e., between any two consecutive zeros of \( u \) there is exactly one zero of \( v \) and vice versa.
20. Consider the equation \( u'' + p(x)u' + q(x)u = 0 \) where \( p \) and \( q \) are continuous on the entire real axis \( R \). Suppose further that \( q(x) < 0 \) there. Show that if \( u \) is not identically zero, it can have at most one zero on \( R \).

21. Let \( u \) and \( v \) be given \( C^2 \) functions on an interval \([a, b]\) whose Wronskian nowhere vanishes there. Show that there is a differential equation of the form (2.7) for which \( u \) and \( v \) form a basis of solutions.

22. Consider the initial-value problem on the interval \([-1, 1]\)

\[
u'' + q(x)u = 0, \quad u(-1) = 1, \quad u'(-1) = 0,
\]

where the coefficient \( q \) is the sectionally continuous function

\[
q(x) = \begin{cases}
-1 & \text{if } -1 \leq x \leq 0, \\
+1 & \text{if } 0 < x \leq 1.
\end{cases}
\]

Find a function \( u \) satisfying the initial conditions, \( C^1 \) on the interval \([-1, 1]\), and satisfying the equation on the intervals \([-1, 0]\) and \((0, 1]\).

23. Suppose \( p \) and \( q \) are continuous functions on the symmetric interval \([-a, a]\) and satisfy the further conditions

\[
p(-x) = -p(x) \quad \text{and} \quad q(-x) = q(x)
\]

there; i.e., \( p \) is an odd function and \( q \) is even. For the differential equation \( u'' + p(x)u' + q(x)u = 0 \) show

(a) If \( u(x) \) is a solution then so is \( v(x) = u(-x) \).

(b) There exists a basis of solutions \( u_1, u_2 \) of which one is even on the interval, the other odd.

2.1.3 The inhomogeneous equation

The study of the inhomogeneous problem

\[
Lu \equiv u'' + p(x)u' + q(x)u = r(x) \tag{2.20}
\]

begins with the observation in Lemma 2.0.2 that implies that if \( u \) and \( v \) are solutions of equation (2.20), then \( w = u - v \) is a solution of equation (2.7). This leads to the notion of a particular integral \( U \), which is any solution of equation (2.20), without regard to initial data.

**Theorem 2.1.8** Let \( U \) be any solution of the inhomogeneous equation (2.20), and let \( u_1, u_2 \) be a basis for the corresponding homogeneous equation (2.7). Then any other solution of (2.20) must be of the form \( U + c_1u_1 + c_2u_2 \), where \( c_1 \) and \( c_2 \) are constants.
Proof: If $u$ is any solution of equation (2.20) then $u - U$ is a solution of the homogeneous equation (2.7) and is therefore a linear combination of $u_1$ and $u_2$. This is the most general solution of equation (2.20). □

In order to determine the constants $c_1$ and $c_2$, further information is needed. If this information is supplied in the form of initial data, this then determines $c_1$ and $c_2$ uniquely.

**Example 2.1.6** Consider the initial-value problem

$$u'' + u = 1, \quad u(0) = 2, \quad u'(0) = -1.$$  

We obtain the particular integral $U(x) = 1$ by inspection. Then $u = 1 + c \cos x + d \sin x$ for some constants $c$ and $d$. Differentiating gives $u' = -c \sin x + d \cos x$. Evaluating these at $x = 0$ now gives $2 = 1 + c$ and $-1 = d$, so the solution is $u = 1 + \cos x - \sin x$.

**Variation of parameters**

In the preceding example, we found a particular integral by inspection. This is of course not always possible and a more systematic way of constructing particular integrals is needed. There is indeed such a construction. It is based on an assumed knowledge of a basis of solutions of the homogeneous equation. We begin by presenting this procedure as a kind of recipe, and discuss the motivation for that recipe at the end this section.

We assume as known a basis $u_1, u_2$ of solutions of the homogeneous equation (2.7). We attempt to represent a particular integral $U$ of equation (2.20) in the form

$$U(x) = c_1(x)u_1(x) + c_2(x)u_2(x),$$

(2.21)

i.e., as if it were a solution of equation (2.3), but allowing the “constants” $c_1$ and $c_2$ to be functions instead. Differentiating this leads to the (rather large) expression

$$U' = c_1'u_1 + c_2'u_2 + c_1u_1' + c_2u_2'.$$

Here we pause and observe that we have introduced two unknown functions, $c_1$ and $c_2$, for the purpose of expressing one, $U$. We should therefore be able to impose one condition on these two functions, and we choose it in the form

$$c_1'u_1 + c_2'u_2 = 0,$$

(2.22)
thereby simplifying the preceding equation by eliminating the first two of the four terms. Now differentiating a second time, we obtain

\[ U'' = c'_1u'_1 + c'_2u'_2 + c_1u''_1 + c_2u''_2. \]

Finally forming the combination \( LU \), and recalling that \( u_1 \) and \( u_2 \) satisfy the homogeneous equation, we obtain, on requiring that \( LU = r \),

\[ c'_1u'_1 + c'_2u'_2 = r. \] (2.23)

Equations (2.22) and (2.23) represent a linear system for the derivatives \( c'_1, c'_2 \); the determinant of this system is the Wronskian \( W(x) \), and therefore nowhere zero. We can now solve this linear system for these derivatives:

\[ c'_1(x) = -u_2(x)r(x)/W(x), \quad c'_2(x) = u_1(x)r(x)/W(x). \]

The solutions, with \( c_1 \) and \( c_2 \) vanishing at a point \( x_0 \) of the interval, are

\[ c_1(x) = -\int_{x_0}^{x} \frac{u_2(s)}{W(s)} r(s) \, ds, \quad c_2(x) = \int_{x_0}^{x} \frac{u_1(s)}{W(s)} r(s) \, ds. \]

The particular integral vanishing together with its first derivative at \( x_0 \) is therefore

\[ U(x) = \int_{x_0}^{x} \left\{ \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1, u_2; s)} \right\} r(s) \, ds. \] (2.24)

We can write this somewhat more compactly by defining the influence function\(^3\). Let \( x_0 = a \) and write

\[ G(x, s) = \begin{cases} (u_1(s)u_2(x) - u_2(s)u_1(x))/W(u_1, u_2; s) & \text{if } s < x, \\ 0 & \text{if } s \geq x \end{cases}. \] (2.25)

Then equation (2.24) takes the form

\[ U(x) = \int_{a}^{b} G(x, s) r(s) \, ds. \] (2.26)

One can now verify that the expression for \( U \) is indeed a solution of equation (2.20). The verification is easiest when the form given in equation (2.24) is used.

\(^3\)This is sometimes called a Green’s function, but we reserve that designation for the analogous function arising in boundary-value problems.
We can equally characterize the influence function by the following initial-value problem for $G$ on the square $[a, b] \times [a, b]$ for fixed $s$ and variable $x$:

$$G(x, s) = 0 \text{ if } s \geq x;$$
$$LG = 0 \text{ for } a \leq s < x \leq b, \text{ and}$$
$$G(s, s) = 0 \text{ and } G_x(s, s) = 1. \tag{2.27}$$

It can be verified (cf. Problem 5 below) that the function defined by equation (2.25) indeed satisfies this initial-value problem.

**Influence function**

The variation-of-parameters procedure could be applied equally in the case $n = 1$ and would immediately provide a particular integral (the second term in (1.23) of the preceding chapter). There, in Section 1.2.3, we indicated that that particular integral is the result of a general principle called Duhamel’s principle and is a natural consequence of the linearity. This is true of the formula (2.24) as well, although this is not transparent in the present case. It becomes transparent when we introduce the system formulation of linear, differential equations in §2.3 below, because that formulation mimics the structure of the first-order equation and the application of Duhamel’s principle becomes obvious.

**PROBLEM SET 2.1.2**

1. For the equation $u'' + u = x$, find a particular integral by inspection. What is the most general solution? What is the solution with initial data $(u, u') = (0, 0)$ at $x = 0$?

2. Same question as above but for the equation $u'' - u = x$.

3. The inhomogeneous form of the self-adjoint equation (cf. Problem Set 2.1.1, exercise 17) is

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u = r(x).$$

In it, the coefficient of the highest derivative is not one, but $p$. Express a particular integral of this equation in terms of a solution basis $u_1, u_2$ for the homogeneous equation and the function $r$, as in equation (2.24) above.

4. Carry out the construction of the influence function for the equation $u'' + u = r$.

5. Verify that the expression for the influence function given in equation (2.25) satisfies the initial-value problem (2.27).
6. Verify directly that if \( G(x, \xi) \) satisfies the initial-value problem (2.27), then
the expression (2.26) provides a particular integral.

7. Consider the homogeneous equation \( u'' + q(x)u = 0 \) where
\[
q(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x \geq 0
\end{cases}
\]
Require of solutions that they be \( C^1 \) for all real values of \( x \) (i.e., continuous
with continuous derivatives: the second derivative will in general fail to exist
at \( x = 0 \)). Construct a basis of solutions. Check its Wronskian.

8. For \( q \) as in Problem 7, find the solution of the initial-value problem
\[
u'' + q(x)u = 1, \quad u(0) = u'(0) = 0.
\]

9. For the differential equation of Problem 8, find the influence function \( G(x, s) \).

10. Let \( L \) be the second-order operator of equation (2.5) characterized by coefficients \( p \) and \( q \) on an interval \([a, b]\), and let \( M \) be a second such operator on
\([a, b]\):
\[
Mv(x) = v'' + r(x)v' + s(x)v.
\]
Assume all the coefficient functions \( p, q, r, s \) are \( C^2 \) on \([a, b]\). Consider the
fourth-order operator \( LM \), where \( LMw \) is formed by applying \( M \) to \( w \) and
then \( L \) to the resulting function \( u = Mw \). Define a basis \( w_1, w_2, w_3, w_4 \) for
this fourth-order operator in terms of bases \( u_1, u_2 \) for \( L \) and \( v_1, v_2 \) for \( M \).
2.2 The equation of order n

The results of the preceding section carry over to higher-order linear equations with remarkably few modifications. The linear operator $L$ is now given by equation (2.1), and the homogeneous equation by equation (2.3). As in the case of the second-order equation, we consider first the existence theorem for the initial-value problem

$$Lu \equiv u^{(n)}(x) + a_1(x)u^{(n-1)}(x) + \cdots + a_{n-1}(x)u'(x) + a_n(x)u(x) = r,$$

$$u(x_0) = \alpha_1, u'(x_0) = \alpha_2, \ldots, u^{(n-1)}(x_0) = \alpha_n,$$  \hspace{0.5cm} (2.28)

where $x_0$ is any point of the interval $[a,b]$ and the numbers $\alpha_1, \ldots, \alpha_n$ are arbitrarily chosen constants.

**Theorem 2.2.1** Suppose the functions $a_1, a_2, \ldots, a_n$ and $r$ are continuous on $[a,b]$. Then the initial-value problem (2.28) has a unique solution $u(x)$ on that interval.

Remarks

- The existence result is again borrowed from Chapter 6.
- The remarks following theorem (2.1.1) apply here as well.
- The uniqueness result is incorporated in this theorem instead of being stated separately. Its proof may be carried out in the same way as in the second-order case except that the definition of $\sigma$ in the proof of Lemma (2.1.1) is changed to

$$\sigma = u^2 + u'^2 + \cdots + u^{(n-1)}^2,$$

and the manipulations are somewhat more cumbersome.
- Theorem 2.1.4 carries over without change (see Problem 10 in Problem Set 2.3.1 below).

It is again convenient to discuss the homogeneous equation first.

2.2.1 The homogeneous equation

Since in place of two arbitrary constants in the solution of a second-order equation we now anticipate $n$ arbitrary constants in the solution of an $n$-th
order equation, we must also anticipate that \( n \) linearly independent solutions are needed to represent an arbitrary solution of the equation (2.3). Accordingly, we define the Wronskian of \( n \) functions on an interval \([a, b]\) as follows:

\[
W(u_1, u_2, \ldots, u_n; x) = \begin{vmatrix}
  u_1(x) & u_2(x) & \cdots & u_n(x) \\
  u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x)
\end{vmatrix}.
\] (2.29)

This definition is valid for any \( n \) functions defined and sufficiently differentiable on \([a, b]\). When these functions are solutions of equation (2.3), the Wronskian serves the same purpose that it did in the second-order case, by virtue of the following theorem:

**Theorem 2.2.2**

\[
W(x) = W_0 \exp \left( - \int_{x_0}^x a_1(s) \, ds \right), \text{ where } W_0 = W(x_0). \quad (2.30)
\]

Remarks:

- This theorem takes exactly the same, simple form for a linear, homogeneous differential equation of any order: the coefficient of the term \( u^{(n-1)} \) in equation (2.1) is what appears in the exponential.

- Corollary 2.1.1 applies almost without change to the Wronskian (2.29) for the \( n \)-th order case:

**Corollary 2.2.1** *The Wronskian of a set of \( n \) solutions of a linear, homogeneous equation (2.3) of order \( n \) either vanishes identically on the entire interval \([a, b]\) or does not vanish at any point of the interval.***

- The proof of Theorem 2.2.2 depends on a formula for the differentiation of determinants; it is discussed in the Problem Set 2.3.1 below.

If we choose a matrix \( A \) with \( ij \) entry \( \alpha_{ij} \), i.e.,

\[
A = \begin{pmatrix}
  \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
  \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix},
\]
we can construct \( n \) solutions \( u_1, u_2, \ldots, u_n \) of equation (2.3) whose Wronskian, at a chosen point \( x_0 \) of \([a, b]\), is precisely \( \det A \). It suffices to choose \( u_k \) to be the solution of equation (2.3) such that

\[
    u_k(x_0) = \alpha_{1k}, \quad u'_k(x_0) = \alpha_{2k}, \ldots, \quad u^{n-1}_k(x_0) = \alpha_{nk}, \quad k = 1, 2, \ldots, n.
\]

These solutions exist by the existence theorem 2.2.1. They are linearly independent on the interval \([a, b]\) if we choose \( A \) so that \( \det A \neq 0 \). This can be done in many ways.

**Example 2.2.1** Choose \( A \) to be the unit matrix, i.e. \( \alpha_{ij} = 0 \) if \( i \neq j \) and \( \alpha_{ii} = 1 \) for \( i, j = 1, \ldots, n \). Then it is easy to see that \( \det A = 1 \).

Theorem 2.1.7 is fundamental. It asserts that once a set of solutions of equation (2.3) that have nonzero Wronskian have been found, any other solution of that equation is a linear combination of them. We now have the same result in general:

**Theorem 2.2.3** A necessary and sufficient condition for a set \((u_1, u_2, \ldots, u_n)\) of solutions of equation (2.3) to be a basis for that equation is that its Wronskian \( W(u_1, u_2, \ldots, u_n; x) \) not vanish on \([a, b]\).

**Example 2.2.2** Consider the third-order equation \( u''' = 0 \). Three linearly independent solutions are \( u_1 = 1, u_2 = x \) and \( u_3 = x^2 \). It is easy to see that any solution of the equation must have the form \( u = c_1u_1 + c_2u_2 + c_3u_3 \).

**Example 2.2.3** Consider the third-order equation \( u''' + u' = 0 \). If \( v = u' \) the equation becomes \( v'' + v = 0 \), which has linearly independent solutions \( v_1 = \cos x \) and \( v_2 = \sin x \). Choose \( u_1 \) such that \( u'_1 = \cos x \). Then \( u_1 = \sin x + \text{constant} \). Similarly choosing \( u_2 \) such that \( u'_2 = \sin x \), we find that \( u_2 = -\cos x + \text{constant} \). Since \( \sin x, \cos x \) and any constant are solutions of this equation, we easily see that \( \{1, \cos x, \sin x\} \) constitute a basis of solutions.

### 2.2.2 The inhomogeneous equation

The two methods used to produce a particular solution of the inhomogeneous equation can be carried out with minor changes. The variation-of-parameters expression is now

\[
    u(x) = \sum_{i=1}^{n} c_i(x) u_i(x)
\]
where \( \{ u_i \} \) represent a basis of solutions of the homogeneous equation. It can be supplemented by \( n - 1 \) conditions on the as-yet-unknown functions \( \{ c_i \} \). These are chosen in the form

\[
\sum_{i=1}^{n} c_i^{(j)} u_i = 0, \quad j = 0, 1, \ldots, n - 2,
\]

where the convention \( u^{(0)} = u \) has been used. The condition that \( u \) satisfy equation (2.4) is then that

\[
\sum_{i=1}^{n} c_i^{(n-1)} = r.
\]

These give \( n \) equations for the \( n \) functions \( c_i' \). The determinant of this system of equations is the Wronskian, so it can be solved for the derivatives of the \( \{ c_i \} \). If they are all integrated from (say) \( x_0 \) to an arbitrary point \( x \) of the interval with the assumption that \( c_i(x_0) = 0 \), then the particular integral \( u \) found in this way will vanish together with its first \( n - 1 \) derivatives at \( x_0 \).

The formula is now more cumbersome to write, so we omit it.

The alternative approach to a particular integral via an influence function is the following. Define the influence function \( G(x,s) \) for \( a \leq s \leq x \leq b \) by the following initial-value problem, where \( s \) is viewed as fixed and the derivatives are taken with respect to \( x \):

\[
L G = 0, \quad s < x \leq b; \quad G(s,s) = 0, \quad \frac{\partial}{\partial x} G(s,s) = 0, \ldots, \frac{\partial^{n-2}}{\partial x^{n-2}} G(s,s) = 0,
\]

\[
\frac{\partial^{n-1} G}{\partial x^{n-1}} (s,s) = 1.
\] (2.31)

On the remainder of the square \([a,b] \times [a,b]\) where \( s > x \), \( G \) vanishes. It can be verified directly that this provides a particular integral of equation (2.4) in the form

\[
u(x) = \int_{a}^{x} G(x,s) r(s) \, ds.
\] (2.32)

We’ll return repeatedly in this book to the single equation of order \( n \), but next we’ll have a first look at an important reformulation: the system of \( n \) first-order equations.

### 2.3 The first-order system

We introduce in this section a somewhat different formulation of linear differential equations. In fact, the formulation applies to nonlinear differential
equations as well, but we restrict our present considerations to the linear case. We begin by considering the \( n \)-th order linear equation (2.4).

### 2.3.1 The system equivalent to the \( n \)-th order equation

There is an alternative representation of the \( n \)-th order linear equations (2.3) and (2.4); we’ll concentrate on the latter since it contains the former if \( r \equiv 0 \). Define

\[
v_1(x) = u(x), v_2(x) = u'(x), \ldots, v_n(x) = u^{(n-1)}(x).
\]

The equation (2.4) is now equivalent to the system of \( n \) first-order equations

\[
\begin{align*}
v_1' &= v_2, \quad v_2' = v_3, \quad \ldots, \quad v_{n-1}' = v_n, \\
v_n' &= -a_1 v_n - a_2 v_{n-1} - \ldots - a_n v_1 + r.
\end{align*}
\]

If initial-data are provided in the form indicated in equation (2.28), then \( v_i(x_0) = \alpha_i \) for \( i = 1, \ldots, n \).

This has a compact expression in vector-matrix notation. Denote by \( v \) the column vector

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}
\]

and by \( A \) the matrix \(^4\)

\[
A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}.
\]

The inhomogeneous equation then has the expression

\[
v' = Av + R,
\]

where \( v' \) is the vector made up of the corresponding derivatives of the components of \( v \), and

\[
R = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r \end{pmatrix}.
\]

\(^4\)A matrix with this structure is called a companion matrix.
2.3.2 The general, linear system

The form of equation (2.37) looks, perhaps superficially, like that of the general linear equation of first order encountered in the first chapter. It turns out that the resemblance is more than superficial. Procedures from the abstract (like the existence theorem) to the practical (like numerical solutions) are unified by this formulation. Moreover, there is no need to restrict the first-order systems of \( n \) equations that we consider to those derived from a single, \( n \)-th order equation: we are free to consider the more general case of equation (2.37) above wherein the matrix \( A = (a_{ij}(x)) \) is an arbitrary matrix of continuous functions, rather than one having the special companion-matrix structure of equation (2.36) above, and the vector \( R = (r_1(x), \ldots, r_n(x))^t \) is an arbitrary vector of continuous functions rather than having the special structure of equation (2.35).

Consider again the initial value-problem

\[
\frac{dv}{dx} = A(x)v + R(x), \quad v(x_0) = v_0,
\]

where \( A(x) \) is an \( n \times n \) matrix whose entries are continuous functions on an interval \( I \), \( R(x) \) is an \( n \)-component vector whose components are continuous functions on \( I \), \( x_0 \) is a point of \( I \), and \( v_0 \) is an arbitrary vector of constants.

We again appeal to Chapter 6 for the basic conclusion:

**Theorem 2.3.1** The initial-value problem (2.39) possesses a unique solution on \( I \).

This means that there is a vector \( v \) whose components are differentiable functions of \( x \) on \( I \), satisfies the differential equation \( v' = Av + R \) at each point of \( I \), and reduces to the given vector \( v_0 \) when \( x = x_0 \). Again, as in the case of the single equation of order \( n \), it is useful to consider the homogeneous case \( R \equiv 0 \) first.

The homogeneous system

The homogeneous system is written in our current vector notation as

\[
\frac{dv}{dx} = A(x)v.
\]

If \( v \) and \( w \) are both vector solutions of this equation, so also is \( \alpha v(x) + \beta w(x) \) for arbitrary constants \( \alpha \) and \( \beta \). This leads as before to the notion of linear dependence of vector-valued functions on an interval \( I \):
Definition 2.3.1 The \( k \) \( n \)-component vector functions \( v_1(x), \ldots, v_k(x) \) are said to be linearly dependent on the interval \( I \) if there exist \( k \) real numbers \( c_1, \ldots, c_k \), not all zero, such that
\[
c_1 v_1(x) + \cdots + c_k v_k(x) = 0
\] 
(2.41)
at each point \( x \) of \( I \). Otherwise they are said to be linearly independent.

In the theory of finite-dimensional vector spaces (like \( \mathbb{R}^n \), for example), a set of \( k \) \( n \)-component vectors \( u_1, \ldots, u_k \) is likewise said to be linearly dependent if a relation \( c_1 u_1 + \cdots + c_k u_k = 0 \) holds for scalars \( \{c_i\}_1^k \) that are not all zero. The vectors \( v_1, \ldots, v_k \) above are functions on an interval \( I \) and belong, for any particular value \( x_0 \) in \( I \) to the \( n \)-dimensional vector space \( \mathbb{R}^n \). But the requirement of linear dependence is more stringent under the definition 2.3.1 than the corresponding definition for a finite-dimensional vector space. The relation
\[
c_1 v_1(x_1) + \cdots + c_k v_k(x_k) = 0
\] 
might hold for many values \( x_j \) with a set of constants that are not all zero (i.e., they may be linearly dependent as vectors in \( \mathbb{R}^n \) at those points) without implying linear dependence on \( I \): for this the equation above would have to hold, for a fixed set of constants \( c_1, c_2, \ldots, c_k \) at every point of \( I \).

For example, consider the vector functions \( v_1(x) = (x,x)^t \) and \( v_2(x) = (x,-x)^t \) on the interval \([-1,1]\) (the superscript \( t \) stands for transpose). It is easy to see that these are linearly independent under the definition 2.3.1. However, if we set \( x = 0 \), each of them is the zero vector and they are linearly dependent as vectors in \( \mathbb{R}^2 \).

It is clear from the definition that, for any set of \( k \) functions \( \{v_1(x), \ldots, v_k(x)\} \) defined on the interval \( I \), linear dependence on \( I \) implies linear dependence on \( \mathbb{R}^n \) for each \( x \in I \). Since linear independence is the negation of linear dependence, we also have linear independence on \( \mathbb{R}^n \) at any point of \( I \) implies linear independence on \( I \).

We can easily infer the existence of a set of \( n \) linearly independent solutions of the homogeneous equation: it suffices to choose them to be linearly independent as vectors in \( \mathbb{R}^n \) at some point \( x_0 \in I \). For example, we could choose \( v_1(x_0) = (1,0,\ldots,0)^t \) to be the first unit vector, \( v_2(x_0) = (0,1,\ldots,0)^t \) to be the second unit vector, and so on. We now show that the solutions \( v_1(x), v_2(x), \ldots, v_n(x) \) of the homogeneous equation with these initial data are linearly independent according to the definition 2.3.1. For suppose that, for some value \( x_1 \in I \), the \( n \) vectors
\(v_1(x_1), v_2(x_1), \ldots, v_n(x_1)\) are linearly dependent as vectors in \(R^n\). Then there are constants \(c_1, \ldots, c_n\), not all zero, such that

\[
c_1 v_1(x_1) + \cdots + c_n v_n(x_1) = 0.
\]

Using these constants, denote by \(v(x)\) the sum

\[
v(x) = c_1 v_1(x) + \cdots + c_n v_n(x).
\]

It is a solution of the homogeneous equation vanishing at \(x_1\). But by the uniqueness theorem it must vanish identically on \(I\). This is not possible at \(x_0\) unless \(c_1 = \cdots = c_n = 0\), a contradiction. Since \(x_1\) could have been any point of \(I\) and therefore, by the remark above, linearly independent on \(I\). We have proved the following theorem:

**Theorem 2.3.2** Let each of the \(n\)-component vector functions \(\{v_j(x)\}_{1}^{n}\) satisfy the homogeneous differential equation (2.40), and suppose that for some \(x_0 \in I\) the vectors \(\{v_j(x_0)\}_{1}^{n}\) are linearly independent as vectors in \(R^n\). Then \(\{v_j(x)\}_{1}^{n}\) are linearly independent on \(I\). If, on the other hand the vectors \(\{v_j(x_0)\}_{1}^{n}\) are linearly dependent as vectors in \(R^n\), then \(\{v_j(x)\}_{1}^{n}\) are linearly dependent on \(I\).

As before, a set of linearly independent solutions of the homogeneous equation is called a basis of solutions, and any solution of the homogeneous equation can be written as a linear combination of them. If \(v_1, \ldots, v_n\) is a basis of solutions, one can form the matrix

\[
\Phi(x) = \begin{pmatrix}
(v_1)_1(x) & \cdots & (v_n)_1(x) \\
\vdots & \ddots & \vdots \\
(v_1)_n(x) & \cdots & (v_n)_n(x)
\end{pmatrix}.
\]  

(2.42)

This matrix, sometimes called a \textit{fundamental matrix} solution of equation (2.40), and consisting of columns of a basis of vector-valued solutions, satisfies the matrix version of the latter equation,

\[
\frac{d\Phi}{dx} = A(x)\Phi,
\]  

(2.43)

and is nonsingular at each point \(x\) of \(I\). A useful version of this matrix is that which reduces to the identity matrix \(E\) at a specified point \(x_0\). In terms of this matrix, the solution \(v(x)\) of the homogeneous problem reducing to the vector \(v_0\) when \(x = x_0\) is \(v(x) = \Phi(x)v_0\).
The Inhomogeneous Solution

Return now to the inhomogeneous problem (2.39), and suppose that we have solved the homogeneous problem and can form a fundamental matrix solution $\Phi$, a nonsingular solution of the matrix-differential equation

$$\frac{d\Phi(x)}{dx} = A(x)\Phi(x), \ a \leq x \leq b.$$  \hfill (2.44)

Duhamel's principle now provides a particular integral for the inhomogeneous equation, in the following way. Denote by $\Phi(x, s)$ the solution of equation (2.44) satisfying the initial condition $\Phi(s, s) = I$. Then the particular integral $P(x)$ may be written (see §1.2.3 of Chapter 1)

$$P(x) = \int_a^x \Phi(x, s)R(s) \, ds,$$  \hfill (2.45)

as is easily verified. It is clear that if $\Phi(x)$ is any fundamental matrix solution, then $\Phi(x, s) = \Phi(x) \Phi^{-1}(s)$.

Let us return now to the variation-of-parameters procedure for a single equation of order two, as described in §2.1.3. We can derive this procedure from equation (2.45) in the following way. Reduce the equation of order two to a system of two first-order equations via equations (2.34), leading to the companion matrix $A(x)$ (equation (2.36 with $n = 2$). If $u$ and $v$ are a basis of solutions for the original, second-order equation, then

$$\Phi(x) = \begin{pmatrix} u(x) & v(x) \\
 u'(x) & v'(x) \end{pmatrix}$$

is a fundamental, matrix solution of equation (2.44). Use this in formula (2.45) (recall that $R(x) = (0, r(x))^t$). This leads to the variation-of-parameters solution of the inhomogenous equation. Of course this is not confined to the case $n = 2$ but can be extended for arbitrary values of $n$.

**PROBLEM SET 2.3.1**

1. Work out the Wronskian for the solutions found in Example 2.2.2. Referring to Theorem 2.2.2, explain why the Wronskian is constant.

2. Same problem as the preceding for the solutions of Example 2.2.3.

3. Find a basis of solutions for the system $u'' + u'' = 0$. Calculate the Wronskian and check the result against Theorem 2.2.2 (equation 2.30).
4. Same as Problem 3 but for the equation

\[ u^{(iv)} + u'' = 0. \]

5. Write out the Wronskian \( W(u_1, u_2, u_3) \) for three functions. Assuming that they satisfy a third-order, linear, homogeneous differential equation, derive the formula of Theorem 2.2.2.

6. The formula for the derivative of a determinant whose entries depend on a variable \( x \) is

\[
\frac{d}{dx} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a'_{11} & \cdots & a'_{1n} \\ \vdots & \ddots & \vdots \\ a'_{n1} & \cdots & a'_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a'_{n1} & \cdots & a'_{nn} \end{vmatrix},
\]

that is, it consists of \( n \) determinants, of which the first is obtained by differentiating the entries in the top row and leaving the others unchanged, the second by differentiating the entries of the second row and leaving the others unchanged, etc. Using this and other, more familiar, rules for manipulating determinants, prove the general version of Theorem 2.2.2.

7. For the operator \( Lu = u''' + u' \), find the most general solution of the equation \( Lu = 1 \) (cf. Example 2.2.3).

8. For the operator of the preceding problem, obtain the influence function for solving the inhomogeneous problem.

9. Carry out the uniqueness part of Theorem 2.2.1 if \( n=3 \).

10. Carry out the proof of Theorem 2.1.4 for the \( n \)th-order homogeneous problem.

11. Find the equivalent first-order system (that is, find the matrix \( A \) and the vector \( R \) of equation (2.37)) for the second-order equation

\[ u'' + x^2 u' + x^4 u = \frac{1}{1 + x^2}. \]

12. In the general, inhomogeneous system \( v' = A(x)v + R(x) \), choose \( \Phi \) to be a fundamental matrix solution of the homogeneous problem that reduces to the identity at \( x = x_0 \). Make the substitution (variation of parameters!) \( v(x) = \Phi(x)w(x) \) to find a particular integral \( v \) that vanishes at the point \( x_0 \) of \( I \).
13. Let $|\Phi(x)|$ represent the determinant of an $n \times n$ matrix solution of equation (2.43) above. Using the formula given in problem 6 above, show that $|\Phi(x)|$ satisfies a first order equation

$$\frac{d|\Phi|}{dx} = a(x)|\Phi|,$$

and express the function $a(x)$ in terms of the entries of the matrix $A$.

14. Carry out the program outlined in the last paragraph of this chapter to derive the variation-of-parameters formula for the case of a single, second-order equation.
Bibliography


