Chapter 5

Isolated singular points

We take up in this chapter a classical subject in the theory of linear differential equations. We shall consider differential equations that are analytic except at isolated singular points. A singular point z_0 is isolated if there is a neighborhood of z_0 in which it is the only singular point. We concentrate mostly on second-order equations, but some of the features emerge already for first-order equations of the form

$$w' + p(z)\, w = 0 \tag{5.1}$$

where p has an isolated singularity at the origin. This isolated singularity is said to be *regular* if p has at most a first-order pole there.

5.1 The Euler equation

The prototype of equations with a regular singular point is the Euler equation,

$$z^2 w'' + \alpha z w' + \beta w = 0, \tag{5.2}$$

where α and β are constants. It is clear, on dividing through by z^2 in order to rewrite this equation in standard form, that the coefficients are then analytic in a neighborhood of the origin except at the origin itself. The method of the last chapter of substituting a power-series for the solution must therefore fail in general, and a new method of expressing the solution in a neighborhood of the origin must be sought. This, as it happens, is extremely easy: substitute $w = z^\mu$. Then the left-hand side becomes

$$\{\mu(\mu - 1) + \alpha\mu + \beta\}\, z^\mu.$$

This indeed vanishes, for all $z \neq 0$, provided the *indicial equation*

$$\mu^2 + (\alpha - 1)\,\mu + \beta = 0 \tag{5.3}$$

is satisfied. If there are two distinct roots μ_1, μ_2 to this equation, there will be two linearly independent solutions $w_1 = z^{\mu_1}, w_2 = z^{\mu_2}$.

What happens if the roots are not distinct? Then $w_1 = z^{\mu_1}$ remains a solution but, since $\mu_2 = \mu_1$, a second solution must be sought otherwise. We can do this with the aid of the Wronskian formula

$$w_2(z) = w_1(z) \int_{z_0}^{z} \frac{W(\zeta)}{w_1(\zeta)^2} d\zeta, \tag{5.4}$$

where z_0 is some appropriately chosen point ($z_0 \neq 0$). Since p, the coefficient of w', is αz^{-1}, we find

$$W(z) = \exp\left(-\int_{z_0}^{z} \alpha\zeta^{-1} d\zeta\right) = K z^{-\alpha},$$

where $K \neq 0$ is a constant. Exploiting the fact that, if μ_1 is a double root of the indicial equation then $\mu_1 = (1-\alpha)/2$, we find from the Wronskian formula that the expression for w_2 is

$$w_2(z) = z^{\mu_1} \int_{z_0}^{z} K\zeta^{-\alpha}\zeta^{\alpha-1} d\zeta = K z^{\mu_1} \ln z - K \ln z_0 z^{\mu_1}. \tag{5.5}$$

We summarize this:

Theorem 5.1.1 *The Euler equation (5.2) has the linearly independent pair of solutions z^{μ_1}, z^{μ_2} if the indicial equation has distinct roots μ_1, μ_2, or $z^{\mu_1}, z^{\mu_1} \ln z$ if there is a single root of multiplicity two.*

The indices μ_1 and μ_2 are obtained as the roots of a quadratic equation, and can be complex even if that equation has real coefficients. In order to make sense of the function z^μ for complex μ we recall the general definition $z^\mu = \exp \mu \ln z$. Since when $z = re^{i\theta}$ we have $\ln z = \ln r + i\theta$, there are various branches of $\ln z$ depending on the choice of interval for $(\theta_0, \theta_0 + 2\pi)$ for θ, and there will be corresponding branches of z^μ in general. When necessary to avoid ambiguity, we shall specify the branch. If no specification is made, it can be assumed that $0 \leq \theta < 2\pi$. If we write $\mu = \nu + i\sigma$ in terms of its real and imaginary parts, we can write the function z^μ in the form

$$z^\mu = r^\nu e^{-\sigma\theta} e^{i(\nu\theta + \sigma \ln r)}.$$

PROBLEM SET 5.1.1

1. Verify directly that $w_2 = z^{\mu_1} \ln z$ is a solution of the Euler equation (5.2) when μ_1 is a double root.
2. Show that, for any complex constant μ, the functions z^μ and $z^\mu \ln z$ are linearly independent over the complex numbers.
3. In the Euler equation (5.2) introduce the change of variable $t = \ln z$ and obtain Theorem 5.1.1 from the known properties of the resulting equation.
4. For the real equation $x^2u'' - xu' + 2u = 0$ on the positive real axis, find a basis of real solutions.
5. The same as the preceding problem for the equation $x^2u'' - xu' + u = 0$.
6. Find the influence function for the differential equation
$$u'' + 2x^{-1}u' - 2x^{-2}u = r(x)$$
on the interval $[1, \infty]$ of the real axis.
7. In the first-order equation (5.1) suppose that p has a pole of order one at $z = 0$ and is otherwise analytic and single valued. Show that the solution has the structure
$$w\,(z) = z^c \sum_{k=0}^{\infty} a_k z^k \tag{5.6}$$
of a (possibly) multivalued factor times a convergent power series.
8. In equation (5.1) put $p(z) = 1/z^2$ and verify that the solution $w(z)$ cannot have the form given in equation (5.6).

5.2 The circuit matrix

Consider the case of an isolated singularity – not necessarily regular! The equation is
$$Lw \equiv w'' + p(z)w' + q(z)w = 0, \tag{5.7}$$

where p and q will be assumed to be analytic in a punctured disk D_0: analytic throughout a disk except at the origin (it is therefore single-valued in D_0). If we pick a point $z_0 \neq 0$ that lies in D_0, we can find a basis of solutions in a disk δ_0 with z_0 at its center; the radius of δ_0 must be less than $|z_0|$. We can find a linearly independent pair of solutions $w_1(z), w_2(z)$ in δ_0. Choose a point z_1 in δ_0 and form a second disk δ_1 overlapping δ_0, throughout which the coefficients are analytic. There exist functions $\tilde{w}_1, \tilde{w}_2$ in δ_1 representing extensions of w_1, w_2 to the union of the two disks. To find $\tilde{w}_1$, pick a point z_1 lying in the intersection of the disks and solve the initial-value problem on δ_2 using the initial data $w_1(z_1), w_1'(z_1)$. Similarly extend w_2. Continuing in this way along a circular path surrounding the origin, we arrive again in the interior of δ_0 with two functions $u_1(z), u_2(z)$ (say), which we can indicate with the peculiar notation $u_k(z) = w_k(ze^{2\pi i})$[1]. We necessarily have

$$w_1(ze^{2\pi i}) = a_{11}w_1(z) + a_{12}w_2(z), \quad w_2(ze^{2\pi i}) = a_{21}w_1(z) + a_{22}w_2(z).$$

These coefficients $\{a_{ij}\}$ define the circuit matrix A.

Proposition 5.2.1 *Show that the matrix A must be nonsingular.*

The proof is left to the reader.

Maybe we picked a 'bad' linearly independent set w_1, w_2. If v_1, v_2 is a second linearly independent set, and

$$w_1 = S_{11}v_1 + S_{12}v_2, \quad w_2 = S_{21}v_1 + S_{22}v_2,$$

this would result in exactly the same linear combinations giving $w_k(ze^{2\pi i})$ in terms of $v_k(ze^{2\pi i})$ and we can write, for the vectors

$$v(z) = \begin{pmatrix} v_1(z) \\ v_2(z) \end{pmatrix}, \quad w(z) \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}$$

the equation

$$V(ze^{2\pi i}) = S^{-1}ASV(z).$$

We choose S to diagonalize A if possible (always so if the eigenvalues of A are distinct). Then

$$v_1(ze^{2\pi i}) = \lambda_1 v_1(z),$$

[1]Denoting the function $f(z)$ as $F(r, \theta)$, what we mean by $f(ze^{2\pi i})$ is $F(r, \theta + 2\pi)$. Thus, for example, if $f(z) = z^\mu = e^{\mu \ln z}$, $f(ze^{2\pi i}) = z^\mu e^{2\pi i \mu}$.

and similarly for v_2. Define α_1 by

$$\lambda_1 = e^{2\pi i \alpha_1},$$

and put

$$f_1(z) = z^{-\alpha_1} v_1(z).$$

The function f_1 is analytic in the punctured disk except for the possible necessity of a branch cut to render it single-valued. But note that

$$f_1(ze^{2\pi i}) = f_1(z)$$

by virtue of the relation between α_1 and λ_1. Thus $v_1(z) = z^{\alpha_1} f_1(z)$ where f_1 is analytic and single-valued in D_0. A similar relation holds for v_2 if the eigenvalues λ_1 and λ_2 are distinct.

What if there is a double eigenvalue? Then it may not be possible do diagonalize the circuit matrix A with a similarity transformation. It is always possible, however, to reduce it to Jordan canonical form, which we may write

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

For $v_1(z)$ we have the same relation as above but now for v_2 we find

$$v_2(ze^{2\pi i}) = \lambda_1 v_2(z) + v_1(z).$$

Define

$$f_2(z) = z^{-\alpha_1} v_2(z) - C \ln z f_1(z),$$

where C is a constant to be determined. Then

$$\begin{aligned} f_2(ze^{2\pi i}) &= z^{-\alpha_1} e^{-2\pi i \alpha_1} \left(\lambda_1 v_2(z) + v_1(z)\right) - C \left(\ln z + 2\pi i\right) f_1(z) \\ z^{-\alpha_1} v_2(z) &+ \lambda_1^{-1} f_1(z) - C \ln z f_1(z) - 2\pi i C f_1(z) = f_2(z) \end{aligned}$$

with the choice $C = (2\pi i \lambda_1)^{-1}$. This shows that the second solution has the structure

$$v_2(z) = z^{\alpha_1} f_2(z) + C z^{\alpha_1} f_1(z) \ln z$$

where f_2, like f_1, is analytic in the punctured disk.

In order to know more about the solution, we need to know more about the nature of the singularity. If the singularity is regular, as it is for an Euler equation, we can say more: the functions f_1 and f_2 appearing above are then not only analytic in a disk about the origin but also bounded there, and may be determined by a power-series solution, as in the next section. If the singularity at the origin is irregular, discovering the nature of the solution may be more complicated. We conclude this section with an example of an irregular singular point, but which can be solved explicitly.

Example 5.2.1 *An irregular singular point*

$$w'' - (\frac{1}{2z} + \frac{1}{z^2})w' + (\frac{1}{2z^2} + \frac{2}{z^3})w = 0.$$

It has solutions

$$w_1(z) = z^{1/2}e^{-1/z} \quad \text{and} \quad w_2(z) = w_1(z)\int_{z_0}^{z} \frac{ds}{w_1(s)}.$$

5.3 The method of Frobenius

The differential equation

$$w'' + p(z)\, w' + q(z)\, w = 0 \tag{5.8}$$

has an isolated singular point at the origin if the coefficients p and q are analytic and single-valued in a disk $|z| < R$ except at $z = 0$ (analytic in the *punctured* disk). The origin is a *regular* singular point if p has a pole of order at most one and q a pole of order at most two there. In other words, if the origin is a regular singular point then $p(z) = z^{-1}P(z)$ and $q(z) = z^{-2}Q(z)$ where P and Q are analytic and single-valued in the full disk, including the origin. We'll write the standard equation with a regular singular point at the origin in the form

$$z^2 w'' + zP(z)\, w' + Q(z)\, w = 0. \tag{5.9}$$

The functions P and Q then have power-series expansions

$$P(z) = \sum_{k=0}^{\infty} P_k z^k, \quad Q(z) = \sum_{k=0}^{\infty} Q_k z^k \tag{5.10}$$

convergent in this disk.

Example 5.3.1 Bessel's differential equation is

$$z^2 w'' + z w' + \left(z^2 - n^2\right) w = 0. \tag{5.11}$$

It arises often in physical problems having an underlying cylindrical symmetry. □

Example 5.3.2 Legendre's differential equation is

$$\left(1 - z^2\right) w'' - 2 z w' + \lambda w = 0. \tag{5.12}$$

This equation has singular points at $z = \pm 1$ rather than at the origin. It arises often in problems having an underlying spherical symmetry. □

If in equation (5.9) the coefficients P and Q were constants, this would be an Euler equation and the method of the previous section would suffice. The method of Frobenius is a generalization of that method. Instead of seeking solutions of the form $w = z^\mu$ (which won't work: see the following Exercises) we seek a solution of the form

$$w\left(z\right) = z^\mu \sum_{k=0}^{\infty} a_k z^k. \tag{5.13}$$

The index μ is as yet undetermined, as is the convergence of the series.

Substitution of the expression above in the differential equation (5.9) now leads, as in the general theory of power series in the preceding chapter, to the conditions

$$I\left(\mu + n\right) a_n + \sum_{k=0}^{n-1} \left\{\left(\mu + k\right) P_{n-k} + Q_{n-k}\right\} a_k = 0 \tag{5.14}$$

where I denotes the indicial polynomial

$$I\left(t\right) = t\left(t - 1\right) + P_0 t + Q_0. \tag{5.15}$$

Equation (5.14) continues to hold for $n = 0$ on the understanding that the sum vanishes in that case. This gives $I\left(\mu\right) a_0 = 0$. There is no point in allowing a_0 to vanish. If $a_0 = 0$ but $a_1 \neq 0$, then the sum (5.13) takes the form $z^{\mu+1}\left(a_1 + a_2 z + \cdots\right)$. Writing $\mu + 1 = \nu$, $a_1 = b_0, a_2 = b_1, \ldots$, we would then find the condition $I\left(\nu\right) b_0 = 0$. If we were to assume that $a_1 = 0$ but $a_2 \neq 0$, we would proceed similarly. Some coefficient a_k must eventually be nonzero, and μ is left undetermined until we decide which. There is therefore no loss of generality in assuming $a_0 \neq 0$. This provides the indicial equation

$$I(\mu) \equiv \mu(\mu - 1) + P_0\mu + Q_0 = 0. \tag{5.16}$$

In order for a solution of the kind given by equation (5.13) to be possible, it is necessary that the index μ satisfy equation (5.16). We have to check that this condition is sufficient to define all the coefficients, and we have to determine the radius of convergence of the series. We turn first to the former, and consider the latter only in the next section.

Suppose that μ satisfies the indicial equation. We can then assign a_0 arbitrarily (but not zero) and seek to find subsequent coefficients on the basis of the recursion formula (5.14). It is clear that, as long as $I(\mu + n) \neq 0$, this formula indeed determines all subsequent coefficients. $I(\mu + n)$ can only vanish if μ and $\mu + n$ are the two roots of the quadratic polynomial I, i.e., the two roots would have to differ by an integer. Even if this is so, if we select the root with the larger real part, there is no breakdown in the recursion formula and all coefficients are determined. If the two roots are the same, it remains true that the recursion formula determines the coefficients. We therefore conclude

> There is always at least one formal solution in the form given by equation (5.13). If the roots of the indicial equation do not differ by an integer, there are two formal solutions of this form.

We expect that, when there are two solutions in this form, they are linearly independent (see following Exercises).

Suppose that the roots do indeed differ by an integer. Let the real part of μ_1 be greater, or at least not less, than that of μ_2: $\mu_2 = \mu_1 - N$ where N is a non-negative integer. Then the indicial equation implies that

$$P_0 = 1 - 2\mu_1 + N, \quad Q_0 = \mu_1(\mu_1 - N). \tag{5.17}$$

One solution of equation (5.9) is of the form $w_1(z) = z^{\mu_1} f_1(z)$ where f_1 is expressed as the power series $\sum a_k z^k$. We can once again invoke the Wronskian formula (5.4) to investigate the nature of the second solution. The Wronskian may be written

$$W(z) = z^{-P_0}\psi(z) \tag{5.18}$$

where ψ is analytic and single-valued in any disk in which the power-series part of P converges. The integrand in the Wronskian formula is therefore

$$\frac{\zeta^{-P_0}\psi(\zeta)}{\zeta^{2\mu_1} f_1(\zeta)^2}.$$

There is nothing to prevent the function f_1 from having zeros, but it cannot vanish at the origin since $a_0 \neq 0$. It therefore cannot vanish in a sufficiently small disk containing the origin and we shall confine the discussion to such a small disk[2]. The function ψ/f_1^2 may then be assumed analytic and single-valued there, and consequently has a power-series expansion. The integrand may now be written

$$\zeta^{-2\mu_1-P_0}\left(c_0 + c_1\zeta + \cdots + c_k\zeta^k + \cdots\right) = \zeta^{-N-1}\sum_{k=0}^{\infty} c_k\zeta^k,$$

where we have used equation (5.17). Term-by-term integration now gives

$$c_n \ln z + z^{-N}\sum_{k=0}^{\infty} d_k z^k,$$

where the coefficients $\{d_k\}$ are related to the $\{c_k\}$ in a simple way. This provides a second solution

$$w_2(z) = c_n w_1(z)\ln z + z^{-N} w_1(z)\sum_{k=0}^{\infty} d_k z^k.$$

Recalling the structure of w_1 and the relation $\mu_1 - N = \mu_2$, we find that we can write

$$w_2(z) = z^{\mu_2} f_2(z) + C w_1(z)\ln z, \tag{5.19}$$

where $f_2(z) = \sum_{k=0}^{\infty} b_k z^k$, i.e., is given by a formal power series. The recursion relation for the coefficients $\{b_k\}$ can be obtained by substitution of the expression above for w_2 into equation (5.9).

We summarize what we have learned about finding formal solutions valid near a regular singular point:

[2]This is a matter of convenience rather than of necessity; the structure of the solution thus found holds in a disk of a known size, as is discussed in the next section.

Lemma 5.3.1 *The equation (5.9) possesses a pair of formal solutions w_1 and w_2. Either they both have the structure of equation (5.13) or one (w_1) has this structure and w_2 has the structure given by equation (5.19); this can only happen in the exceptional case when the zeros of the indicial equation differ by an integer.*

PROBLEM SET 5.3.1

1) Suppose that in equation (5.9) the coefficients P and Q are *not* both constants. Show that this equation *cannot* have a basis of solutions $w_1(z) = z^{\mu_1}$, $w_2(z) = z^{\mu_2}$ where μ_1 and μ_2 are complex constants.
2) Derive the conditions (5.17) when the roots of the indicial polynomial differ by an integer N.
3) Derive in detail the conditions (5.14).
4) Suppose there are two solutions in the form of equation (5.13) with indices μ_1 and μ_2 that do not differ by an integer. Assume further that the power series converge in a neighborhood of the origin. Show that these two solutions are linearly independent there.
5) Find the indicial equation for Bessel's equation (5.11), and find the indices. For the case when n is a non-negative integer, obtain the series solution for the index with maximum real part. Do the indices differ by an integer?
6) Consider the singular points of Legendre's equation (5.12). Are they regular? Find the indicial equation and the indices relative to any regular singular point.
7) Suppose the coefficients of equation (5.8) are analytic and single-valued in a punctured neighborhood of the origin. Suppose it is known that the function $w(z) = f(z) \ln z$ is a solution, where f is analytic and single-valued in the punctured neighborhood. Deduce that f is also a solution.
8) Lamé's equation,

$$\begin{aligned} & \left(a_1^2 + t\right)\left(a_2^2 + t\right)\left(a_3^2 + t\right) u'' \\ + \quad & \frac{1}{2}\left[\left(a_1^2 + t\right)\left(a_2^2 + t\right) + \left(a_2^2 + t\right)\left(a_3^2 + t\right) + \left(a_3^2 + t\right)\left(a_1^2 + t\right)\right] u' \\ & = (At + B)\, u, \end{aligned} \tag{5.20}$$

arises in problems in ellipsoidal domains. The numbers a_1, a_2, a_3, A, B are all real constants. Locate the regular singular points of Lamé's equation and find the indicial equation and the indices for one of them (your choice).

9) Find the recursion relation for the coefficients $\{b_k\}$ in the power-series expansion of the function f_2 appearing in equation (5.19). What determines the constant C?

5.4 Convergence

The solutions above remain formal until we show that the power-series factors converge. The power series expansions (5.10) converge within some disk of radius R, say, which is the distance from the origin to the nearest singularity (other than that at $z = 0$) of either of the coefficients p or q. The power-series factors in the formal solutions in fact converge with this same radius of convergence R. We shall carry out the proof of this first for a solution of the kind given in equation (5.13), i.e., we shall show that the series $\sum a_k z^k$ converges. Then we turn briefly to the exceptional case when the second solution involves a logarithm[3].

We begin with a pair of elementary results, the first regarding the behavior of the indicial polynomial (5.15):

Proposition 5.4.1 *There exists a positive number B, as large as we please, and a second positive number $A > B$, such that*

$$|I\,(k+\mu)| \geq (k-A)\,(k+B)\,.$$

Here μ is a root of I and k is any non-negative integer.

The proof is left as an exercise.

Proposition 5.4.2 *Let $\{c_k\}_0^\infty$ be a sequence of non-negative numbers satisfying*

$$c_k \leq \lambda\,(1 + \max\{c_0, c_1, \ldots, c_{k-1}\}) \tag{5.21}$$

where $0 < \lambda < 1$. Then $\{c_k\}_0^\infty$ is bounded.

Proof: If not, then for an increasing sequence $\{M_j\}$ with $M_j \to \infty$, there is an increasing sequence k_j of integers such that $c_{k_j} \geq M_j$. We can choose the sequences such that $c_k < M_j$ if $k < k_j$. Choose j large enough so that $M_j > \lambda/(1-\lambda)$, and use $k = k_j$ in equation (5.21):

$$M_j \leq c_{k_j} < \lambda(1 + M_j),$$

[3] A very complete and general version of the proof is given in [?], Appendix B.

implying $M_j < \lambda/(1-\lambda)$, a contradiction. □

We now estimate the coefficients a_k. Since we are assuming the indices do not differ by an integer, $I\left(k+\mu\right)$ does not vanish and we have, by equation (5.14),

$$|a_k| \le \frac{1}{|I\left(k+\mu\right)|}\sum_{l=0}^{k-1}\left\{\left(l+|\mu|\right)|P_{k-l}|+|Q_{k-l}|\right\}|a_l|\,.$$

If $\rho < R$, the series $\sum|P_j|\,\rho^j$ and $\sum|Q_j|\,\rho^j$ converge and therefore, if $R_j = \max\left(|P_j|\,,|Q_j|\right)$, so also does $\sum R_j\rho^j$. Together with the Proposition above this implies

$$|a_k| \le \frac{1}{k-A}\sum_{l=0}^{k-1}\frac{l+|\mu|+1}{k+B}R_{k-l}\,|a_l| \le \frac{1}{k-A}\sum_{l=0}^{k-1}R_{k-l}\,|a_l|\,,$$

where we have chosen $B > |\mu|$. Therefore

$$|a_k|\,\rho^k \le \frac{1}{k-A}\sum_{l=0}^{k-1}R_{k-l}\rho^{k-l}\,|a_l|\,\rho^l \le \frac{\mathrm{Max}\left\{|a_l|\,\rho^l\right\}}{k-A}\sum_{l=0}^{k-1}R_{k-l}\rho^{k-l},$$

where the maximimum is taken over $l = 0,1,\ldots,k-1$. Because of the convergence of the series $\sum R_j\rho^j$ we find that $\sum_{l=0}^{k-1}R_{k-l}\rho^{k-l} \le \sum_0^\infty R_j\rho^j = S$ (say). Therefore there is a sufficiently large integer K such that, for $k \ge K$,

$$|a_k|\,\rho^k \le \lambda\max\left\{|a_l|\,\rho^l\right\},$$

where the maximum is again taken over $l = 0,1,\ldots,k-1$ and λ may be chosen in the interval $(0,1)$. This shows by Proposition 5.4.2 that the sequence $\left\{|a_k|\,\rho^k\right\}$ is bounded. Therefore, for any z such that $|z| < \rho$, $\sum a_k z^k$ converges. Since ρ was any number less than R, it follows easily that the radius of convergence of $\sum a_k z^k$ is at least R.

Let the second solution be written $w_2 = u + C(\log z)w_1(z)$ where $u(z) = z^{\mu_2}\sum b_k z^k$, and what is at issue is the convergence of the series part of u, and therefore the estimation of the coefficients $\{b_k\}$. Substitution of this expression into the differential operator, noting that w_1 is a solution, and requiring that w_2 also be a solution, leads to the recursion formula

$$I(\mu_2+k)b_k = -\sum_{l=0}^{k-1}\left(P_{k-l}((l+\mu_2)+Q_{k-l}\right)b_l$$
$$-C\left\{\left(2k-2N+2\mu_1-1+P_0\right)a_{k-N}+\sum_{i=0}^{k-N-1}P_{k-N-i}a_i\right\}$$

where $N = \mu_1 - \mu_2$ and the term in curly braces is interpreted as zero if $k < N$, and as $(2\mu_1 - 1 + P_0)a_0$ if $k = N$. The first term on the right is estimated as above and we find

$$\begin{aligned}
|b_k|\rho^k &\leq \frac{1}{k-A}\sum R_{k-l}\rho^{k-l}|b_l|\rho^l \\
&+ \frac{C\rho^k}{(k-A)(k+B)}\left\{(2k+2\mu_1-2N-1+P_0)|a_{k-N}| + \sum_{i=1}^{k-N-1} R_{k-N-i}|a_i|\right\} \\
&\leq \frac{1}{k-A}\sum R_{k-l}\rho^{k-l}|b_l|\rho^l \\
&+ \frac{C\rho^N}{(k-A)}\left\{|a_{k-N}|\rho^{k-N} + \sum_{i=1}^{k-N-1} R_{k-N-i}\rho^{k-N-i}|a_i|\rho^i\right\}
\end{aligned}$$

where we have used a factor of $k + B$ to overcome the the factor of $|a_{k-N}|$. Since the series $\sum a_k z^k$ is already known to have radius of convergence not less than R, we have $|a_{k-N}|\rho^{k-N} < M$ for some constant M. Similarly in the last sum above, which is estimated by $M \sum R_{k-N-i}\rho^{k-N-i} \leq MS$, where S is a bound for $\sum_0^\infty R_k\rho^k$. We now obtain

$$|b_k|\rho^k \leq \frac{1}{k-A}\left(S\max\{|b_0|, \ldots, |b_{k-1}\rho^{k-1}\} + CR^N(M+MS)\right).$$

Choosing k sufficiently large, we find

$$|b_k|\rho^k \leq \lambda\left(1 + \max\{|b_0|, \ldots, |b_{k-1}|\rho^{k-1}\}\right)$$

with $\lambda \in (0,1)$, which implies, by Proposition 5.4.2, that $|b_k|\rho^k$ is bounded, and therefore the series $\sum b_k z^k$ converges for any z with $|z| < R$.

Theorem 5.4.1 *The formal solutions described in Lemma 5.3.1 are in fact solutions; the power series have radii of convergence at least equal to the smaller of the radii of convergence of the power series for P and Q.*

PROBLEM SET 5.4.1

1. Prove Proposition 5.4.1.

2. Consider the third-order equation having a regular singular point at the origin:
$$z^3 w''' + a_1(z) z^2 w'' + a_2(z) z w' + a_1(z) w = 0,$$
where a_1, a_2, a_3 are analytic in a domain including the origin. This can be transformed to a system of three first-order equations, by the definitions $w_1 = w, w_2 = z w_1', w_3 = z w_2'$, of the form $zW' = A(z)W$ where W is the vector with components w_1, w_2, w_3 and A is a three-by-three matrix. Work out A.

3. For Chebyshev's equation
$$(1 - z^2) w'' - z w' + \lambda w = 0,$$
where λ is a constant, locate the regular singular points in the finite complex plane and find the indices relative to these points.

4. Consider the equation
$$(1 - z) z^2 w'' + (z - 4) z w' + 6w = 0.$$
   (a) Verify that this equation has a regular singular point at the origin.
   (b) Find the indicial equation and the indices relative to this point.
   (c) For the index with the greater real part, find the recursion relation for the coefficients in the series solution.
   (d) Determine whether the second solution is given purely by a series solution (as in equation 5.12) or involves in addition a logarithmic term (as in equation (5.18).

The following four problems relate to singular points at infinity. These are investigated by making the transformation $t = 1/z$ and investigating the singular points at $t = 0$. In each case determine whether the point in question is a point of analyticity, a regular singular point, or an irregular singular point. In the case of a regular singular point, find the indices.

5. Bessel's equation $z^2 w'' + z w' + (n^2 - z^2) w = 0$ where n is a constant.

6. Legendre's equation $(1 - z^2) w'' - 2 z w' + \lambda w = 0$, where λ is a constant.

7. The equation $w'' + w = 0$.

8. The equation $z^2 w'' + w = 0$

5.5 The system formulation

The general case of a regular singular point is handled most neatly in the system formulation,

$$zw'(z) = A(z)w, \tag{5.22}$$

where w is an n-component, complex-valued vector, and A is a matrix-valued function of position analytic in a neighborhood of the origin. The single, nth order, linear, homogeneous equation with a regular singular point at the origin,

$$z^n v^{(n)} + P_1(z) z^{n-1} v^{(n-1)} + \cdots + P_n(z) v = 0,$$

can be placed in this form with the aid of the definitions

$$w_1 = v, \quad w_2 = zw_1', \quad w_3 = zw_2', \quad \ldots, \quad w_n = zw_{n-1}'.$$

Example 5.5.1 *The case* $n = 2$ Here we put $w_1 = v, \quad w_2 = zv'$ and obtain

$$zw' = A(z)w, \quad A(z) = \begin{pmatrix} 0 & 1 \\ -P_2(z) & 1 - P_1(z) \end{pmatrix}.$$

Consider, however, the more general case of equation (5.22) with the usual assumption that A is analytic in a neighborhood of the origin. Here the distinction, alluded to above, between the behavior of the equation and that of solutions becomes more important. Denote the singularity of the equation as *Fuchsian*, and reserve the designation *regular* for solutions: a solution $w(z)$ is regular if $z^k w(z)$ remains bounded in a full neighborhood of the origin, for some choice of the complex constant k. The singular point $z = 0$ is regular if all solutions are regular in this sense. One of the consequences of the theory of such systems is that Fuchsian singular points are necessarily regular. However, not all regular singular points need be Fuchsian.

Example 5.5.2 Consider the two-dimensional system

$$w' = z^{-2} Z w, \quad Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The equations for the components are $w_1' = z^{-2} w_2$ and $w_2' = 0$, with the solution $w_2 = c_2, w_1 = c_1 - c_2 z^{-1}$, where c_1 and c_2 are constants. These solutions are regular and therefore so is the singular point, but the latter is not Fuchsian.