Chapter 8

Stability I: Equilibrium Points

Suppose the system

$$\dot{x} = f(x), \quad x \in R^n \tag{8.1}$$

possesses an equilibrium point q i.e., $f(q) = 0$. Then $x = q$ is a solution for all t. It is often important to know whether this solution is *stable*, i.e., whether it persists essentially unchanged on the infinite interval $[0, \infty)$ under small changes in the initial data. This is particularly important in applications, where the initial data are often known imperfectly. Below we give a precise definition of stability for equilibrium solutions of systems of differential equations, and this chapter is devoted to this subject. The system 8.1 is *autonomous*, i.e., the vector function f has no explicit dependence on the independent variable.

We restrict consideration to *Lyapunov* stability, wherein only perturbations of the initial data are contemplated, and thereby exclude consideration of *structural* stability, in which one considers perturbations of the vector field (cf. [?] for a discussion of structural stability).

8.1 Lyapunov Stability

Consider the system (8.1). We assume that f is in $C^1(\Omega)$ where Ω is a domain in R^n. We denote by $x = \phi(t, p)$ the solution of equation (8.1) taking the value p when $t = 0$, noting that in the autonomous case there is no loss of generality in taking the initial instant $t_0 = 0$ (see Problem 2 of Problem Set 6.4.1 above).

Definition 8.1.1 *The equilibrium point q is said to be* stable *if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(t,p) - q\| < \epsilon$ for all $t > 0$ and for all p such that $\|p - q\| < \delta$. If δ can be chosen not only so that the solution q is stable but also so that $\phi(t,p) \to q$ as $t \to \infty$, then q is said to be* asymptotically stable. *If q is not stable it is said to be* unstable.

In the following example the origin of coordinates is an equilibrium point, and there may be other equilibrium points as well.

Example 8.1.1 The following system of three equations, the so-called Lorenz system, arose as a crude model of fluid motion in a vessel of fluid heated from below (like a pot of water on a stove). It is a widely studied (see [**?**]) example of simple dynamical system in which chaotic behavior may occur, although tame behavior like equilibrium solutions may also occur. The system is

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz. \tag{8.2}$$

Here the numbers σ, r and b are constants. The numbers σ and b are usually regarded as fixed, positive numbers, whereas different values for the constant r – which measures the vigor of the external heating – may be considered. □

If $\phi(t,p) = q + \xi(t,p)$, then equation (8.1) takes the form

$$\dot{\xi} = A\xi + \dots, \quad \text{where} \quad A = f_x(q) \tag{8.3}$$

where $f_x(q)$ denotes the Jacobian matrix evaluated at q, i.e., its ij entry is $\partial f_i/\partial x_j$. The dots indicate terms vanishing faster than linearly in ξ. It seems natural, therefore, to try to relate the stability of the solution to that of the zero solution of the linear equation

$$\dot{\xi} = A\xi. \tag{8.4}$$

This strategy, we shall find in §8.4 below, meets with considerable success, and we begin with a study of equation (8.4). The question of the stability or instability of the solution $\xi = 0$ of the *linear* problem (8.4) will be called the *linearized stability* problem. The matrix A can be any matrix with real entries.

Equation (8.4) is the linear system with constant coefficients studied in Chapter 3, §3.6, so we shall make several references below to this section.

When $n = 1$ the system (8.4) reduces to the one-dimensional equation $\dot{x} = ax$ with solution $x = \exp(at)x_0$. Thus the origin is stable if $a \leq 0$ and unstable if $a > 0$. This case is so simple that it fails to convey any of the complexity of the general problem. However, when $n = 2$, a number of features of the general problem are captured in a setting that is still quite simple. We therefore consider this case in the next section.

8.2 Linearized stability: $n = 2$

Equation (8.4) can in the present case be written

$$\dot{\xi}_1 = a_{11}\xi_1 + a_{12}\xi_2, \quad \dot{\xi}_2 = a_{21}\xi_1 + a_{22}\xi_2.$$

If the matrix A has eigenvalues λ_1 and λ_2 with corresponding linearly independent eigenvectors $\xi^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)})^t$ and $\xi^{(2)} = (\xi_1^{(2)}, \xi_2^{(2)})^t$[1], then the most general solution is

$$\xi(t) = c_1\xi^{(1)} \exp \lambda_1 t + c_2\xi^{(2)} \exp \lambda_2 t \tag{8.5}$$

where c_1 and c_2 are arbitrary constants. In the present case, the characteristic polynomial is

$$p(\lambda) = \lambda^2 + (a_{11} + a_{22})\lambda + \Delta = 0, \quad \Delta = a_{11}a_{22} - a_{12}a_{21}.$$

Since we are assuming that A is a real matrix, this polynomial has real coefficients and either its roots are both real or they are complex conjugates: $\lambda_{1,2} = \rho \pm i\sigma$, where ρ and σ are real. In the latter case, the eigenvectors ξ_1 and ξ_2 are likewise complex conjugates and for the solution (8.5) to be real the complex constants c_1 and c_2 are also complex conjugates.

8.2.1 The case when both eigenvalues are real

If the eigenvalues are both negative, then the solution clearly decays to zero exponentially and the origin is not only stable but also asymptotically stable. If one of the eigenvalues is zero and the other is negative, then the origin is stable but not asymptotically stable. On the other hand, if (at least) one

[1] The superscript 't' stands for transpose, indicating that we think of the vectors $\xi^{(1)}$ and $\xi^{(2)}$ as column vectors.

of the eigenvalues is positive, the origin is unstable. For example, suppose $\lambda_1 > 0$. Then, choosing $c_2 = 0$ in equation (8.5), we find that for any nonzero choice of c_1 the norm of the solution increases without bound, implying instability.

A case not covered by these considerations is that when there is only one independent eigenvector. This can happen only when $\lambda_2 = \lambda_1 = \lambda$ (say). This is illustrated by the following example.

Example 8.2.1 Consider the matrix

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

It has the eigenvalue λ with multiplicity two, and the single eigenvector $\xi^{(1)} = (1,0)^t$. To achieve a basis we may define $\xi^{(2)} = (0,1)^t$ and represent arbitrary initial data in the form

$$\xi(0) = c_1\xi^{(1)} + c_2\xi^{(2)}.$$

Direct verification shows that

$$\xi(t) = \left\{(c_1 + c_2 t)\,\xi^{(1)} + c_2\xi^{(2)}\right\}\exp\lambda t$$

is the solution of equation (8.4) taking on the prescribed initial data. It is clear that if $\lambda \geq 0$ then the origin is unstable. If $\lambda < 0$ then, while the term $t\exp\lambda t$ increases initially, it reaches a finite maximum and $\|\xi(t)\|$ can be made arbitrarily small by choosing c_1 and c_2 sufficiently small; this implies not only stability but also asymptotic stability.

8.2.2 The case when the eigenvalues are complex

Here $\lambda = \rho \pm i\sigma$ and we may assume that $\sigma \neq 0$ for otherwise the eigenvalue is real (and of multiplicity two), and is discussed above. We could leave the solution in the form given by equation (8.5) above with the proviso that $c_2 = \overline{c_1}$ for real solutions, but it is instructive to adopt an alternative approach.

Let $\xi^{(1)} = \eta + i\zeta$ where η and ζ are the real and imaginary parts of the complex vector $\xi^{(1)}$. Then (verify this!)

$$A\eta = \rho\eta - \sigma\zeta \text{ and } A\zeta = \sigma\eta + \rho\zeta.$$

Figure 8.1: Two examples of orbits in the x_1x_2-plane in the case when the origin is a stable equilibrium point of the system (8.4): (a) when both eigenvalues are real (and not equal), and (b) when they are complex conjugates with negative real part.

The general solution of equation (8.4) takes the form $\xi(t) = a(t)\eta + b(t)\zeta$ for real, time-dependent functions a, b provided that they satisfy the equations

$$\dot{a} = \rho a + \sigma b, \quad \dot{b} = -\sigma a + \rho b. \tag{8.6}$$

It is easy to see that these have the general solution

$$a(t) = \exp \rho t \left\{a_0 \cos \sigma t + b_0 \sin \sigma t\right\}, \quad b(t) = \exp \rho t \left\{-a_0 \sin \sigma t + b_0 \cos \sigma t\right\}. \tag{8.7}$$

This shows that the origin is stable if $\rho \leq 0$ and asymptotically stable if ρ is strictly negative; it is unstable otherwise.

Figure 8.1 provides pictures of the orbits in asymptotically stable cases.

8.2.3 Canonical forms for matrices

The pictures of the orbits given in Figure 8.1 can easily be generalized to other cases (see for example [?] for a more complete set of diagrams). However, even for the asymptotically stable cases indicated in that figure, the pictures shown are simplified. This simplified form can be obtained by a change of coordinates that brings the matrix A of the system to one of the two forms

$$(a): \ A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } (b): \ A_2 = \begin{pmatrix} \rho & \sigma \\ -\sigma & \rho \end{pmatrix}. \tag{8.8}$$

These are canonical forms for the matrix A and may be arrived at as follows (cf. [?]).

Consider first the case when the eigenvalues λ_1 and λ_2 of A are both real with linearly independent column eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$. Form the matrix S whose columns are $\xi^{(1)}$ and $\xi^{(2)}$, and let S^{-1} be its inverse (how do we know that this exists?). If in equation (8.4) we make the substitution $x = Sy$, we find that y satisfies the linear, differential equation $\dot{y} = A_1 y$ where A_1 is the matrix given in equation (8.8). The orbits of *this* system are shown in Figure 8.1 (a): to find the orbits of the original system one would transform back to the x system resulting in axes that are in general not at right angles to one another and to corresponding distortions of the pictured orbits.

In the exceptional case alluded to above in which both eigenvalues have the same value λ and there is only one linearly independent eigenvector $\xi^{(1)}$, we may introduce a second vector (sometimes called a *generalized eigenvector*) $\xi^{(2)}$ such that $A\xi^{(2)} = \lambda\xi^{(2)} + \xi^{(1)}$. Introducing the transformation matrix S as above now leads to the system $\dot{y} = A_3 y$ where A_3 is the matrix of Example 8.2.1 above.

Finally, in the case of a complex conjugate pair of eigenvalues we introduce the column vectors η and ζ of §8.2.2 and form the matrix S from these. Transforming again via $x = Sy$ we get the equations (8.6) with $y_1 = a, y_2 = b$. The orbits of these equations are those that are shown in Figure 8.1 (b). To get the pictures in the original x system, we again need to transform back, resulting in non-orthogonal axes and distortions of the pictured orbits.

It is clearly useful to pass to coordinate systems in which the equations take simple forms. These are canonical forms that a matrix may take (the *Jordan* canonical form in the cases of matrices A_1 and A_3, the *real* canonical form in the case of the matrix A_2). Below we describe the Jordan canonical form in generality. Its derivation, which is a generalization of that sketched above in the two-dimensional case, is available in many books treating linear algebra (cf. [?],[?]).

8.3 Linear Stability: general values of n

We now consider the equation (8.4) when the constant-coefficient matrix A is of arbitrary dimension n. We will be interested mostly in the case when the matrix A, the dependent variable x and the time t are real, but most of the development below applies also when they are complex. Even when A

and t are restricted to real values, it will be useful to allow x to take complex values.

In §3.6.2 above, we have defined the matrix exponential function $\exp At$ and shown that it is the fundamental matrix solution of equation (8.4); in other words, the solution of that equation taking the initial value x_0 is $x(t) = (\exp At)\, x_0$. It is therefore clear that the stability or instability of the zero solution of equation (8.4) is entirely determined by the behavior of $\exp At$ as $t \to +\infty$.

The matrix, or operator norm, is defined in general by $\|A\| = \sup_{\|x\|=1} \|Ax\|$, in terms of the vector norm[2]. A choice of norm that we will sometimes make below is

$$\|x\| = \sum_{i=1}^{n} |x_i|. \tag{8.9}$$

One can then show that the matrix norm is given by the formula

$$\|A\| = \sup_j \sum_{i=1}^{n} |A_{ij}|. \tag{8.10}$$

Here the notations $|x_i|$ and $|A_{ij}|$ represent the moduli of the corresponding complex numbers.

A transformation $x = Py$ for any constant, nonsingular matrix P may be viewed as a change of coordinates, between $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. If we introduce this transformation into equation (8.4) we find a similiar equation

$$\dot{y} = By, \quad B = P^{-1}AP. \tag{8.11}$$

Thus the matrix B is *similar* to A. This equation has the fundamental matrix solution $\exp Bt$. Since the relation $P^{-1}AP = B$ implies that $P^{-1}A^kP = B^k$ for any non-negative integer k, it follows from the power-series definition of $\exp At$ that $P^{-1}e^{At}P = e^{Bt}$.

We now choose P to reduce A to Jordan canonical form J. We can write J in the following form involving $r + 1$ block matrices $J_k, (k = 0, 1, ..., r)$ along the diagonal:

$$J = \begin{pmatrix} J_0 & 0 & \cdots & 0 \\ 0 & J_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_r \end{pmatrix} \tag{8.12}$$

[2]This is discussed in Chapter 6; see in particular Problem Set 6.2.1, problems 9 and 10.

where J_k is $n_k \times n_k$. Here $J_0 = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_p)$ are eigenvalues (not necessarily distinct) belonging to one-dimensional eigenspaces. To each $k = 1, \ldots, r$, there corresponds an eigenvalue λ_{p+k} and a generalized eigenspace of dimension $n_k \geq 2$. The corresponding Jordan block may be written

$$J_k = \lambda_{p+k} I_{n_k} + Z_{n_k}, \tag{8.13}$$

where I_m is the $m \times m$ identity matrix and Z_m is an $m \times m$ nilpotent matrix of the form

$$Z = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8.14}$$

It is worth remarking here that the Jordan canonical form (8.12) is not unique, since the ordering of the blocks $J_1, \ldots, J_r$ is arbitrary, as is the distribution of the eigenvalues appearing in J_0, since each of the latter belongs to a one-dimensional eigenspace.

Since the block structure of J is preserved under multiplication it follows that

$$e^{Jt} = \begin{pmatrix} e^{J_0 t} & 0 & \cdots & 0 \\ 0 & e^{J_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_r t} \end{pmatrix}. \tag{8.15}$$

Now for any of the matrices J_k, the exponential matrix $e^{J_k t}$ has a simple form. It is particularly simple for J_0 viz.

$$e^{J_0 t} = \text{diag}(e^{\lambda_0 t}, \ldots, e^{\lambda_p t}). \tag{8.16}$$

For $k \geq 1$, according to equation (8.13), J_k is the sum of two matrices which commute because one of them is a multiple of the identity. Hence we find (see Proposition 3.6.1 of Chapter 3)

$$e^{J_k t} = e^{(\lambda_{p+k} I + Z)t} = e^{\lambda_{p+k} t} e^{Zt}. \tag{8.17}$$

i.e., each block beyond the first is the product of a complex number $e^{\lambda_{p+k} t}$ with the exponential of a nilpotent matrix. The latter is easily expressible in terms of the power series expansion, because that series terminates: if $Z = Z_m$, then $Z^n = 0$ if $n \geq m$. An easy calculation now gives

$$e^{Zt} = \begin{pmatrix} 1 & t & \cdots & t^{m-1}/(m-1)! \\ 0 & 1 & \cdots & t^{m-2}/(m-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{8.18}$$

This gives a complete description of the e^{At} in terms of the eigenvalues of A and the structure of the generalized eigenspaces.

Up to the present, whereas we may have thought of t as real, we have not needed this assumption. In the next theorem, however, we do need to assume that t is real.

Theorem 8.3.1 *If Re(λ_k) $\leq \alpha$ for each $k = 1, \ldots, p+r$, then there exists a constant C and and an integer $m \geq 0$ such that, $\forall\, t \geq 0$,*

$$\|e^{At}\| \leq C\left(1 + t + \cdots + \frac{t^m}{m!}\right) e^{\alpha t}. \tag{8.19}$$

Proof: First consider $\|e^{Jt}\|$. From (8.15) we see that this matrix preserves the block structure of J. The norm (8.10) is a supremum over column sums, and by the block structure we see that any column sum includes entries from only a particular block $e^{J_k t}$. Thus it is clear that

$$\|e^{Jt}\| \leq e^{\alpha t}\left[1 + t + \cdots + \frac{t^m}{m!}\right],$$

where the integer m is at most 1 less than the dimension of the largest generalized eigenspace.

Since $P^{-1}e^{At}P = e^{J(t)}$, we have

$$\|e^{At}\| \leq K\|e^{Jt}\| \quad \text{where}\, K \geq \|P\|\|P^{-1}\|. \quad \square$$

Remark: We have referred specifically to the norm (8.10) in the proof of this theorem, but the conclusion is independent of the choice of norm in virtue of the fact that, in finite dimensions, any two norms are equivalent; see §6.2 above.

Corollary 8.3.1 *For any $\beta > \alpha$ and some $K > 0$, $\|e^{At}\| \leq Ke^{\beta t}$.*

Proof: It's only necessary to observe that, for any $\epsilon > 0$, the function $t^k \exp(-\epsilon t)$ has a maximum for positive t. The conclusion follows on putting $\epsilon = \beta - \alpha$. □

This result has the following important special case:

Corollary 8.3.2 *If $Re(\lambda_k) < 0$ for each k, then $\|e^{At}\| \leq Ke^{-\gamma t}$ for some $\gamma > 0$.*

The following conclusions for the stability or instability of the linear system (8.4) of size n are immediate:

Theorem 8.3.2 *The origin is asymptotically stable for the system (8.4) if the real part of every eigenvalue is negative. It is unstable if any eigenvalue has a positive real part.* □

PROBLEM SET 8.3.1

1. Consider the logistic equation, equation (1.26) of Chapter 1. Show that it has two equilibrium solutions and discuss the linearized stability of each.
2. Consider the two-dimensional, linear system $\dot{x} = Ax$ where
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Is the origin stable or unstable?
3. For the Lorenz system of Example 8.1.1 show that the origin is an equilibrium point and discuss its linearized stability.
4. Consider the two-dimensional, linear system $\dot{x} = Ax$ where
$$A = \begin{pmatrix} -\delta & 1 \\ 0 & -\delta \end{pmatrix}$$
where δ is a positive number so that the origin is asymptotically stable. Define the *amplification factor* for a solution $x(t)$
$$a = \sup ||x(t)||/||x(0)||,$$
where the supremum is taken over all $t > 0$ and the norm is the Euclidean norm: $||x|| = \sqrt{x_1^2 + x_2^2}$. Show that $a \geq 1/(\delta e)$ where e is the base of the natural logarithm.

5. Suppose that the origin is an unstable equilibrium point of the system (8.1) according to Definition 8.1.1. Prove that there exists a positive number ϵ_0 and sequences of vectors $\{x_k\}_1^\infty$ and positive, real numbers $\{t_k\}_1^\infty$ such that $\|x_k\| \to 0$ and $t_k \to \infty$ as $k \to \infty$ and $\|\phi(t_k, x_k)\| \geq \epsilon_0$ for each $k = 1, 2, \ldots$.

6. Prove the formula (8.10).

7. Prove the formulas (8.16), (8.17) and (8.18).

8. For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

find the fundamental matrix solution $\Phi(t) = \exp(At)$ of the equation $\dot{x} = Ax$.

For the next two problems the matrix A is

$$A = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 3/2 \end{pmatrix}. \tag{8.20}$$

9. Verify that a fundamental matrix solution of equation (8.4)[3] with A given by equation (8.20) is

$$\Phi(t) = \begin{pmatrix} (1-t/2)e^t & t/2e^t \\ -t/2e^t & (1+t/2)e^t \end{pmatrix}.$$

10. Arrive at the formula of problem 9 by showing that $\exp(At) = \Phi(t)$. (Hint: find the similarity matrix P taking A to its Jordan form, evaluate $\exp(Jt)$, then transform back via P^{-1} to find $\exp(At)$.)

11. Prove Theorem 8.3.2. Under what conditions on the matrix A of the system (**??**) is the origin stable but not asymptotically stable?

12. In the Jordan form, the nilpotent matrices of the form (8.14) can be exchanged for similar matrices in which the 'ones' on the secondary diagonal are replaced by arbitrary nonzero numbers, e.g., for $\nu \neq 0$,

$$Z_\nu = \begin{pmatrix} 0 & \nu & 0 & \cdots & 0 \\ 0 & 0 & \nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8.21}$$

[3] *Verify* means: work out each side of equation (8.4) and check that they are the same.

Prove this.
(Hint: calculate the effect of acting on a Jordan block $B = \lambda I + Z$ of size m with the similarity transformation $R = \text{diag}\ (\nu^{-m}, \nu^{-m+1}, \ldots, \nu^{-1})$.)

8.4 Nonlinear Stability

In this section we show that, under certain additional assumptions, linearized stability is decisive for stability, i.e., the nonlinear terms in equation (8.3) do not contribute to the qualitative determination of stability. The theorems of this section, Theorems 8.4.1 and 8.4.2 are therefore of wide importance and application.

The basic equation (8.1) may, with $x = q + \xi$, be rewritten as

$$\dot{\xi} = A\xi + g(\xi) \tag{8.22}$$

as in equation (8.3), where A has the same meaning as in that equation, and

$$g(\xi) = f(q + \xi) - f(q) - A\xi = f(q + \xi) - A\xi \tag{8.23}$$

represents the nonlinear terms that were represented by dots in equation (8.3). We therefore wish to consider the initial-value problem consisting of equation (8.22) with initial data ξ_0. Since $\xi = 0$ is a solution of equation (8.22) for all t, the choice $\xi_0 = 0$ leads to this (and only this) solution. We therefore consider only $\xi_0 \neq 0$.

We begin by reformulating the integral version of the initial-value problem (as in equation (6.14) above). It is easy to check directly that the integral equation

$$\xi(t) = \exp(At)\xi_0 + \int_0^t \exp\left(A(t-s)\right) g\left(\xi(s)\right)\, ds \tag{8.24}$$

is equivalent to the initial-value problem for the system (8.22).

Under the simple conditions that we shall adopt below, the function g satisfies a simple estimate:

Lemma 8.4.1 *Suppose the function f appearing in equation (8.1) is C^1 on the domain Ω and g is given by equation (8.23). Then for any $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that*

$$\|g(\xi)\| \leq \epsilon_0 \|\xi\| \quad \textit{provided} \quad \|\xi\| \leq \delta_0. \tag{8.25}$$

Proof: For any C^1 function f the identity

$$f(q+\xi) - f(q) = \int_0^1 f_x(q+u\xi)\xi \, du$$

holds, provided $q \in \Omega$ and ξ is sufficiently small; here f_x represents the Jacobian matrix of f. This identity follows from the observation that the integrand is $(d/du)\, f(q+u\xi)$. Therefore

$$g(\xi) = f(q+\xi) - f(q) - A\xi = \int_0^1 \left(f_x(q+u\xi) - A\right) \xi \, du.$$

With $A = f_x(q)$, the matrix-valued function $f_x(q+u\xi) - A$ vanishes if $\xi = 0$ and is continuous there. It follows that its norm can be made arbitrarily small uniformly with respect to u if $\|\xi\|$ is chosen sufficiently small, and this gives the condition (8.25). □

Now consider the case when all the eigenvalues of the matrix A have negative real parts. According to Corollary 8.3.2, there are positive constants K and γ such that $\|\exp(At)\| \le Ke^{-\gamma t}$ for all $t \ge 0$. We shall assume, without loss of generality, that $K \ge 1$. To prove stability in this case, we need to show that, for any given $\epsilon > 0$, $\|\xi(t)\|$ remains less than ϵ provided that $\|\xi_0\| < \delta$ for some sufficiently small δ. In this, we are free to rechoose ϵ to be smaller than the given value. We shall choose it as follows.

In the preceding estimate for g, choose $\epsilon_0 < \gamma/K$. This determines a positive number δ_0. Choose $\epsilon < \delta_0$. This ensures that as long as $\|\xi(t)\| < \epsilon$, we will have $\|g(\xi(t)\| < \epsilon_0$. Now choose $\delta < \epsilon/K$. We now show that if $\|\xi_0\| < \delta$, we must have $\|\xi(t)\| < \epsilon$ for all positve t. Initially this must be so since $\delta < \epsilon$ so it must hold on some interval to the right of the origin. If it should fail for any positive t, then there is a least value, say T, where it first fails. Then

$$\|\xi(T)\| = \epsilon \text{ whereas } \|\xi(t)\| < \epsilon \text{ on } [0,T). \tag{8.26}$$

Consider equation (8.24) for $t \le T$. Taking norms gives

$$\|\xi(t)\| \le K\delta e^{-\gamma t} + \int_0^t Ke^{-\gamma(t-s)}\epsilon_0 \|\xi(s)\| \, ds$$

where we have exploited the inequality (8.26) so as to be able to use the estimate (8.25). Provisionally define

$$u(t) = \|\xi(t)\| e^{\gamma t}.$$

Then the inequality for $\|\xi(t)\|$ takes the form

$$u(t) \leq K\delta + \int_0^t K\epsilon_0 u(s)\, ds.$$

Gronwall's lemma now implies

$$u(t) \leq K\delta \exp\left(K\epsilon_0 t\right)$$

or

$$\|\xi(t)\| \leq K\delta \exp\left(K\epsilon_0 - \gamma\right) t.$$

The exponent is negative by our choice of ϵ_0 so $\|\xi(t)\| < K\delta$ on $[0, T)$ and, in particular, $\|\xi(T)\| \leq K\delta < \epsilon$, contradicitng the assumption that $\|\xi(T)\| = \epsilon$. This proves stability in this case. Indeed, a minor modification of the reasoning shows that the stability is asymptotic and we have

Theorem 8.4.1 *Suppose that in equation (8.1) the function $f \in C^1(\Omega)$ where Ω is a domain in R^n, and suppose that $q \in \Omega$ is an equilibrium point at which all the eigenvalues of the Jacobian matrix f_x have negative real parts. Then q is an asymptotically stable equilibrium point of f.*

We now turn to the case when the real parts of n_1 of the eigenvalues of the matrix $A = f_x(q)$ are positive, $n_1 \geq 1$. This of course implies instability for the linearized problem and we'll show that it likewise implies instability for the full, nonlinear problem. Suppose then that the eigenvalues are divided into two groups, one in which all real parts are positive (say $\lambda_1, \ldots, \lambda_{n_1}$) and a second in which all real parts are negative or zero (say $\lambda_{n_1+1}, \ldots, \lambda_n$), with $n_1 + n_2 = n$. In each of these groups we must allow for multiplicities.

We begin by transforming coordinates so that the matrix A takes a canonical form with two special features. The first is a transformation to a Jordan-canonical matrix $B = P^{-1}AP$ such that the eigenvalues with positive real parts come first; this exploits the arbitrariness of the Jordan form with respect to the ordering of the eigenvalues. This will give B a block structure

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

in which each of the matrices B_1 and B_2 possesses the Jordan canonical structure: eigenvalues appear on the diagonal, and a certain number of constants occur along a secondary diagonal; in B_1 each eigenvalue has a positive real part whereas in B_2 no eigenvalue has a positive real part. The second special feature is that the constants along the secondary diagonal are all equal to ν, a positive parameter that we are free to choose (cf. Problem 8.21 of the preceding problem set).

The transformation of variables $\xi = P\eta$ will in general require that P and η be in C^n, i.e., that their components be complex, although ξ is in R^n. In the transformation of the nonlinear function $h(\eta) = P^{-1}g(P\eta)$ the values of the function h and of its argument η are in C^n, but h is well defined since the argument of g takes only values in R^n. Lemma 8.4.1 has an immediate extension to h and we shall apply it to h with the same notation for the constants ϵ_0, δ_0 indicated in that lemma.

The strategy is to assume that the origin is stable and deduce a contradiction. Denote by σ a positive lower bound for the real parts of the eigenvalues of B_1. We shall in due course choose ϵ_0 and ν sufficiently small in relation to σ. We may assume that δ in the definition of stability is chosen sufficiently small that the norm of the solution remains less than $\epsilon < \delta_0$, so that the estimate of Lemma 8.25 holds for the function h.

The equation (8.22) takes the form

$$\dot{\eta}_1 = B_1\eta_1 + h_1(\eta), \tag{8.27}$$

$$\dot{\eta}_2 = B_2\eta_2 + h_2(\eta), \tag{8.28}$$

where we have written (η_1, η_2) in place of η, splitting the latter in conformity with the block structure of B. We'll use a norm $R = R_1 + R_2$ where

$$R_1 = \sqrt{\sum_1^{n_1} \overline{\eta_{1i}}\eta_{1i}},$$

where the overbar represents the complex conjugate and n_1 is the dimension of η_1. There is a similar expression for R_2. Then

$$\frac{d}{dt}R_1^2 = \sum_{j=1}^{n_1}\left(\overline{\eta_{1j}}\sum_{l=1}^{n_1} B_{1jl}\eta_{1l} + \text{c.c.}\right) + \sum_{j=1}^{n_1}\left(\overline{\eta_{1j}}h_{1j}(\eta) + \text{c.c.}\right)$$

where c.c. means the complex conjugate of the preceding expression. The first of the two terms above, which arises from the linear terms in the equation, can be separated into two parts, one coming from the diagonal part of the matrix B_1 and the other from the secondary diagonal. That coming from the diagonal term exceeds $2\sigma R_1^2$. That coming from the secondary diagonal is at most $2\nu R_1^2$ in magnitude, and therefore exceeds $-2\nu R_1^2$. The final term, arising from the nonlinear term, is easily estimated with the aid of the Schwartz inequality and is seen to exceed $-2\epsilon_0 R_1 (R_1 + R_2)$, provided that $R_1 + R_2$ remains less than δ_0. This gives

$$\frac{dR_1}{dt} \geq (\sigma - \nu - \epsilon_0) R_1 - \epsilon_0 R_2.$$

In a similar way we deduce for R_2 the inequality

$$\frac{dR_2}{dt} \leq (\nu + \epsilon_0) R_2 + \epsilon_0 R_1.$$

Putting these together gives

$$\frac{d}{dt}(R_1 - R_2) \geq (\sigma - 2\nu - 2\epsilon_0) R_1 - 2\epsilon_0 R_2.$$

The choices $\nu < (3/8)\sigma$ and $\epsilon_0 < (1/8)\sigma$, which we are free to make, now ensures that

$$\frac{d}{dt}(R_1 - R_2) \geq (\sigma/2)(R_1 - R_2).$$

Choosing initial data so that $R_1 > R_2$ at $t = 0$ is now seen to imply that $R_1 - R_2$ increases without bound, contradicting the assumption that $R_1 + R_2 < \epsilon$. This proves

Theorem 8.4.2 *Suppose that in equation (8.1) the function $f \in C^1(\Omega)$ where Ω is a domain in R^n, and suppose that $q \in \Omega$ is an equilibrium point at which at least one eigenvalue of the Jacobian matrix f_x has a positive real part. Then q is an unstable equilibrium point of f.*

The two theorems of this section may be combined into one:

Theorem 8.4.3 *Suppose the C^1 system 8.1 possesses the equilibrium point p and put $T = f_x(p)$. The origin is asymptotically stable if the real part of every eigenvalue of T is negative. It is unstable if any eigenvalue of T has a positive real part.* □

Comparing this with Theorem 8.3.2, we see that linearized stability is decisive for nonlinear stability under the hypotheses of these theorems.

Theorem 8.4.3 covers a lot of ground and is among the most widely quoted theorems in dynamical-systems theory. However, there are patches of ground that it does not fully cover. We explore an interesting such patch in the next section.

8.5 Conservative Systems

The autonomous system (8.1) will be called conservative if there exists a C^1 scalar function $E : \Omega \to R$ which is not constant on any open set in Ω, but is constant on orbits. The function E is called an *integral*, or a *constant of the motion*, of the system (8.1).

Theorem 8.5.1 *An equilibrium point q of a conservative system cannot be asymptotically stable.*

Proof: Suppose q is an asymptotically stable equilibrium point. Then there is a neighborhood N of q such that if $p \in N$, $\phi(t, p) \to q$ as $t \to \infty$. However, if the system is assumed conservative with integral E, then E is constant on orbits, so $E(p) = E[\phi(t, p)] = E(q)$ for each $p \in N$. This implies that E is constant on an open set, which is a contradiction. □

This shows that proving stability for a conservative system cannot rely on Theorem 8.4.1 and will require different methods from those employed there.

When a conservative system has an equilibrium point at which the integral has a minimum, an inference of stability can be made on this basis. In particular

Definition 8.5.1 *A function $E : \Omega \to R$ is said to have a* strong *minimum at q if there is a neighborhood N of q such that $E(x) > E(q)$ for every $x \in N$ except for $x = q$.*

Theorem 8.5.2 *Suppose q is an equilibrium point of a conservative, autonomous system and that its integral E has a strong minimum there. Then q is stable.*

Proof: Let $V(x) = E(x) - E(q)$. Given $\epsilon > 0$ consider the set $\{x : \|x\| = \epsilon\}$. This lies in the neighborhood N if ϵ is small enough and we may assume this is so. Denote by V_ϵ the minimum of V on the set $\|x\| = \epsilon$. Choose $\delta < \epsilon$ such that $V(x) < V_\epsilon$ if $\|x\| < \delta$. Then $\|x(t)\| < \epsilon$ for all $t > 0$. □

The same conclusion can be drawn if E has a strong maximum at equilibrium.

Hamiltonian dynamics is a formulation of the dynamics of point masses with wide applicability. Let q and p be n-vectors (coordinates and momenta, respectively). Hamilton's canonical equations are:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Here the Hamiltonian function $H = H(q, p)$ is a function of $2n$ variables and the system is said to have n degrees of freedom. It follows easily from the equation that if, as indicated here, H is not explicitly dependent on time, it is constant on orbits. Hence if H is either a maximum or a minimum at equilibrium, that equilibrium is stable.

Example 8.5.1 (Linearized instability need not imply instability) Let $H = p^4 + q^2$. The origin is an equilibrium point. The linearized equations are $dq/dt = 0, dp/dt = -2q$. This has solutions increasing linearly with time, so the origin is unstable for the linearized system. However, the Hamiltonian has a minimum there, so it is stable. □

In Hamiltonian systems, linearized stability of the origin is often characterized by motions of an oscillatory character, and nonlinear stability is sometimes taken for granted if linearized stability holds. This does not follow from Theorem 8.4.1 and is a somewhat risky conclusion, particularly when there can be resonant interactions in the nonlinear system. The following example in two degrees of freedom shows this explicitly.

Example 8.5.2 (Linearized stability need not imply stability)

$$H = \frac{1}{2}\left[q_1^2 + p_1^2\right] - \left[q_2^2 + p_2^2\right] + \frac{1}{2}p_2\left[p_1^2 - q_1^2\right] - q_1 q_2 p_1.$$

Then the equations are

$$\begin{aligned} \dot{q}_1 &= p_1 + p_1 p_2 - q_1 q_2 & \dot{p}_1 &= -q_1 + q_1 p_2 + p_1 q_2 \\ \dot{q}_2 &= -2p_2 + (1/2)(p_1^2 - q_1^2) & \dot{p}_2 &= 2q_2 + q_1 p_1. \end{aligned}$$

The linearized system can be read off this system; all its solutions are bounded, and it possesses periodic solutions with periods 2π and 4π. However, for arbitrary T a solution to the full system is

$$p_1 = \sqrt{2}\frac{\sin(t-T)}{t-T},\ p_2 = \frac{\sin 2(t-T)}{t-T},\ q_1 = \sqrt{2}\frac{\cos(t-T)}{t-T},\ q_2 = \frac{\cos 2(t-T)}{t-T}.$$

For small t these approximate to solutions of the linearized system, but if $T > 0$ they become unbounded in finite time. □

PROBLEM SET 8.5.1

1. The reasoning leading to the conclusion of asymptotic stability in Theorem 8.4.1 requires a "minor modificaton." Provide it.

   The following three problems relate to the Lorenz system (8.2).

2. Find the equilibrium solutions of the Lorenz system *other* than the origin. Show that they exist as real solutions only for $r > 1$.

3. Formulate the linearized-stability problem for the equilibrium solutions of Problem 2 and write out the characteristic polynomial $p(\lambda)$ for the eigenvalues.

4. Find the roots of the characterstic equation $p(\lambda)$ of Problem 3 explicitly if $r = 1$. Show that the nonzero solutions are stable for values of $r > 1$, at least if $r - 1$ is not too large.
   (Hint: put $r = 1 + \epsilon$ and seek roots $\lambda = \lambda_0 + \lambda_1 \epsilon + \ldots$.)

   The next three problems relate to the Lotka-Volterra system (see Example 7.3.3 of Chapter 7).

5. Find the equilibrium solutions of the Lotka-Volterra system and discuss (a) linearized stability and (b) stability?

6. Show that the Lotka-Volterra system is conservative by obtaining a function $u(x, y)$ that is constant on orbits.
   (Hint: try $u(x, y) = Ax + By + C \ln x + D \ln y$ where A, B, C, D are constants to be determined.)

7. Use the function u of the preceding problem to infer that the nonzero equilibrium point of the Lotka-Volterra system is stable.

   The following two problems relate to the equations of *rigid-body dynamics*, which may be written (cf. Problem 13 of Problem Set 6.2.1)

   $$I_1\dot{\omega}_1 + (I_2 - I_3)\omega_2\omega_3 = 0,\ I_2\dot{\omega}_2 + (I_2 - I_3)\omega_3\omega_1 = 0,\ I_3\dot{\omega}_3 + (I_1 - I_2)\omega_1\omega_2 = 0.$$

   Here $\omega_1, \omega_2, \omega_3$ are the components of the angular-velocity vector along the principal axes of the rigid body, and I_1, I_2, I_3 are positive constants: the moments of inertia about the corresponding axes.

8. Show that, aside from the trivial equilibrium solution $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$, there are three families of equilibrium solutions of the form $(\omega_1, 0, 0), (0, \omega_2, 0), (0, 0, \omega_3)$.

9. Verify that the equilibrium solution $(\omega_1, 0, 0)$ is stable if the corresponding moment of inertia I_1 is either the greatest or the least of the three moments of inertia, but unstable if it is intermediate between I_2 and I_3.

8.6 Invariant Sets and Manifolds

Consider the autonomous system (7.1) on a domain $D \subset R^n$. A set $\mathcal{S}$) in D is said to be invariant for this system if whenever $x_0 \in \mathcal{S}$ the orbit $\phi(t, x_0)$ through x_0 likewise lies in $\mathcal{S}$. This set need not be a smooth manifold, but if it is a manifold $\mathcal{M}$ of dimension m, then it is referred to as an invariant manifold. If the manifold can be represented in the form $x = \Psi(y)$ for $y \in E \subset R^m$, then the invariance can be expressed in the form $D\Psi(y)\dot{y} = f(\Psi(y))$, a differential equation for y. For a solution $y(t)$ of this equation, the orbit lying on M expressed in R^n is $x(t) = \Psi(y(t))$.

Example 8.6.1 *The three-dimensional system*

$$\begin{aligned} \dot{x}_1 &= -\delta x_1 + x2x_3 + x_3^2, \\ \dot{x}_2 &= -\delta x_2 - x_3 - x_3 x_1 + x_3^2, \\ \dot{x}_3 &= -x_3 - x_3(x_1 + x_2), \end{aligned}$$

where δ is a parameter, has the invariant manifold $x_3 = 0$. For, if $x_1(t), x_2(t)$ represents a solution of the two-dimensional system $\dot{x}_1 = -\delta x_1,\ \dot{x}_2 = -\delta x_2$, then $(x_1(t), x_2(t), 0)$ is easily seen to be a solution of the three-dimensional system. By uniqueness, it is the only solution for which $x_3(0) = 0$.

In this example the equation $x_3 = 0$ of the invariant manifold holds globally, but in general the representation $x = \Psi(y)$ may only hold locally.

It is also true in this example that if $x_0 \in \mathcal{M}$ then $x = \phi(t, x_0) \in \mathcal{M}$ for all $t \in R$, and this is what is normally understood under the term 'invariant.' If on the other hand the condition $x_0 \in \mathcal{M}$ implies that $\phi(t, x_0) \in \mathcal{M}$ only for $t \geq 0$, this needs in principal to be indicated in some way. More generally, suppose the orbit through x_0 has the maximal interval of existence (a, b). Then we can at most expect it to lie in $\mathcal{M}$ on this interval. We shall extend the normal definition of invariance as follows:

Definition 8.6.1 *The set $\mathcal{M}$ is invariant for the system (7.1) if, whenever $x_0 \in \mathcal{M}$, $\phi(t, x_0) \in \mathcal{M}$ for all t in the maximal interval of existence of this solution.*

Now let the system (7.1) have an equilibrium point $x = 0$, i.e., $f(0) = 0$ (if there is an equilibrium point at $x = p \neq 0$ we may imagine it shifted to the origin). If that equilibrium point is asymptotically stable, we may define its *basin of attraction* B as the set of $y \in D$ such that $\phi(t, y) \to 0$ as $t \to \infty$, where $\phi(t, y)$ is the solution of (7.1) with initial point y. It is clear that for each $y \in B$ in the maximal interval of existence (a, b) we have $b = +\infty$. It may be that $a = -\infty$, or a may have a real value.

Theorem 8.6.1 *Suppose the dynamical system (7.1) is defined on a domain $D \in R^n$, has a unique solution for each $y \in D$, and that solution is a continuous function of its initial data (cf. Theorem 6.3.1). Let $x = 0$ be an asymptotically stable equilibrium point for this system. Then its basin of attraction is an open, invariant subset of D.*

Proof: There is a neighborhood $N(0)$ of the origin such that, if $q \in N(0)$, then $\phi(t, q) \to 0$ as $t \to \infty$. Suppose now that $y \in B$. Then for some $t_* > 0$ $\phi(y, t_*) = y_* \in N(0)$. By continuity with respect to initial data, there is a neighborhood $N'(y)$ of y such that, if $y' \in N'(y)$, then $\phi(y', t_*) \in N(0)$, and therefore $y' \in B$. This shows that B is open. To see that it is invariant we only need observe that, if $y \in B$ and $y_1 = \phi(t_1, y)$, then the identity $\phi(t, y_1) = \phi(t + t_1, y)$ shows that $y_1 \in B$ as well. □

The basin of attraction might be all of the domain D, that is, it is possible that for any $y \in D$ $\phi(t, y) \to 0$ as $t \to \infty$, as simple examples show. However, if B is a proper subset of D, then the open set B has a boundary ∂B. The latter is likewise invariant:

Theorem 8.6.2 *The boundary ∂B of the basin of attraction is an invariant set.*

Proof: Let y_0 be in ∂B and suppose that, for some t_1, $y_1 = \phi(y_0, t_1) \notin \partial B$. It is not possible for y_1 to lie in B, since the latter is invariant by the preceding theorem, whereas y_0 by assumption does not lie in B. Therefore $y_1 \in C$ where C is the complement of $\overline{B}$, the closure of B. Since C is an open set, there is a neighborhood N_1 of y_1 lying entirely in C. By continuity with respect to initial data, there is a neighborhood N_0 of y_0 such that if $y \in N_0$

then $\phi(y, t_1) \in N_1 \subset C$. But, since $y_0 \in \partial B$, any neighborhood of y_0 contains points of B. Choosing for $y \in N_0$ a point of B, and recalling the invariance of B, we arrive at a contradiction. □

Remark: the maximal interval (a, b) for the existence of the solution $\phi(t, y_0)$ of this theorem *could* be finite. If p is an equilibrium point of the system (7.1) other than the origin, it necessarily lies in B^C, the complement of B, the basin of attraction of O, by the definition of B. If some of the eigenvalues of the linearization $A = Df(p)$ have negative real parts, then the point p will have a stable manifold[4]. It's clear that this stable manifold must likewise belong to B^C. Since $\partial B \subset B^C$, the point p may lie on the basin boundary: suppose this is so. Then it is possible – but not inevitable – for its stable manifold to belong to ∂B as well. Similar remarks apply if we replace an equilibrium point p with another kind of invariant set S (say, a periodic orbit).

PROBLEM SET 8.6.1

1. Verify equation (7.4).

2. Consider the two-dimensional system

$$\dot{x} = -x - x^3, \quad , \dot{y} = -y + y^2.$$

   Find the equilibrium points and check their stability. Solve explicitly to locate the basin of attraction B of the stable equilibrium point. Show that the maximal interval of existence of any solution starting in B is (a, ∞) where a is a (finite) negative number.

[4]Problem: this has not yet been defined!