Chapter 8

Stability I: Equilibrium Points

Suppose the system

$$\dot{x} = f(x), \quad x \in R^n \tag{8.1}$$

possesses an equilibrium point q i.e., $f(q) = 0$. Then $x = q$ is a solution for all t. It is often important to know whether this solution is *stable*, i.e., whether it persists essentially unchanged on the infinite interval $[0, \infty)$ under small changes in the initial data. This is particularly important in applications, where the initial data are often known imperfectly. Below we give a precise definition of stability for equilibrium solutions of systems of differential equations, and this chapter is devoted to this subject. The system 8.1 is *autonomous*, i.e., the vector function f has no explicit dependence on the independent variable.

We restrict consideration to *Lyapunov* stability, wherein only perturbations of the initial data are contemplated, and thereby exclude consideration of *structural* stability, in which one considers perturbations of the vector field (cf. [5] for a discussion of structural stability).

8.1 Lyapunov Stability

Consider the system (8.1). We assume that f is in $C^1(\Omega)$ where Ω is a domain in R^n. We denote by $x = \phi(t, p)$ the solution of equation (8.1) taking the value p when $t = 0$, noting that in the autonomous case there is no loss of generality in taking the initial instant $t_0 = 0$ (see Problem 2 of Problem Set 6.4.1 above).

Definition 8.1.1 *The equilibrium point q is said to be* stable *if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(t,p) - q\| < \epsilon$ for all $t > 0$ and for all p such that $\|p - q\| < \delta$. If δ can be chosen not only so that the solution q is stable but also so that $\phi(t,p) \to q$ as $t \to \infty$, then q is said to be* asymptotically stable. *If q is not stable it is said to be* unstable.

In the following example the origin of coordinates is an equilibrium point, and there may be other equilibrium points as well.

Example 8.1.1 The following system of three equations, the so-called Lorenz system, arose as a crude model of fluid motion in a vessel of fluid heated from below (like a pot of water on a stove). It is a widely studied (see [5]) example of simple dynamical system in which chaotic behavior may occur, although tame behavior like equilibrium solutions may also occur. The system is

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz. \tag{8.2}$$

Here the numbers σ, r and b are positive constants. The numbers σ and b are usually regarded as fixed, positive numbers, whereas different values for the constant r – which measures the vigor of the external heating – may be considered. □

Suppose indeed that equation (8.1) has an equilibrium solution q: $f(q) = 0$. If $p = q + \xi_0$ is an initial value close to q and we put $\phi(t,p) = q + \xi(t,p)$, then equation (8.1) takes the form

$$\dot{\xi} = A\xi + \ldots, \quad \text{where} \quad A = f_x(q) \tag{8.3}$$

where $f_x(q)$ denotes the Jacobian matrix evaluated at q, i.e., its ij entry is $\partial f_i / \partial x_j$. The dots indicate terms vanishing faster than linearly in ξ. It seems natural, therefore, to try to relate the stability of the solution to that of the zero solution of the linear equation

$$\dot{\xi} = A\xi. \tag{8.4}$$

This strategy, we shall find in §8.4 below, meets with considerable success, and we begin with a study of equation (8.4). The question of the stability or instability of the solution $\xi = 0$ of the *linear* problem (8.4) will be called the *linearized stability* problem. The matrix A can be any matrix with real entries.

Equation (8.4) is the linear system with constant coefficients studied in Chapter 3, §3.6, so we shall make several references below to this section.

When $n = 1$ the system (8.4) reduces to the one-dimensional equation $\dot{x} = ax$ with solution $x = \exp(at)x_0$. Thus the origin is stable if $a \le 0$ and unstable if $a > 0$. This case is so simple that it fails to convey any of the complexity of the general problem. However, when $n = 2$, a number of features of the general problem are captured in a setting that is still quite simple. We therefore consider this case in the next section.

8.2 Linearized stability: $n = 2$

Equation (8.4) can in the present case be written

$$\dot{\xi}_1 = a_{11}\xi_1 + a_{12}\xi_2, \quad \dot{\xi}_2 = a_{21}\xi_1 + a_{22}\xi_2.$$

If the matrix $A = (a_{ij})$ has eigenvalues λ_1 and λ_2 with corresponding linearly independent eigenvectors[1] $\xi^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)})^t$ and $\xi^{(2)} = (\xi_1^{(2)}, \xi_2^{(2)})^t$, then the most general solution is

$$\xi(t) = c_1\xi^{(1)} \exp \lambda_1 t + c_2\xi^{(2)} \exp \lambda_2 t \tag{8.5}$$

where c_1 and c_2 are arbitrary constants. In the present case, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + \Delta = 0, \quad \Delta = a_{11}a_{22} - a_{12}a_{21}.$$

Since we are assuming that A is a real matrix, this polynomial has real coefficients and either its roots are both real or they are complex conjugates: $\lambda_{1,2} = \rho \pm i\sigma$, where ρ and σ are real. In the latter case, the eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$ are likewise complex conjugates and for the solution (8.5) to be real the complex constants c_1 and c_2 are also complex conjugates.

8.2.1 The case when both eigenvalues are real

If the eigenvalues are both negative, then the solution clearly decays to zero exponentially and the origin is not only stable but also asymptotically stable. If one of the eigenvalues is zero and the other is negative, then the origin is stable but not asymptotically stable. On the other hand, if (at least) one of the eigenvalues is positive, the origin is unstable. For example, suppose $\lambda_1 > 0$. Then, choosing $c_2 = 0$ in equation (8.5), we find that for

[1]The superscript 't' stands for transpose, indicating that we think of the vectors $\xi^{(1)}$ and $\xi^{(2)}$ as column matrices.

any non-zero choice of c_1 the norm of the solution increases without bound, implying instability.

A case not covered by these considerations is that when there is only one independent eigenvector. This can happen only when $\lambda_2 = \lambda_1 = \lambda$ (say) and is illustrated by the following example.

Example 8.2.1 Consider the matrix

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

It has the eigenvalue λ with multiplicity two, and the single eigenvector $\xi^{(1)} = (1, 0)^t$. To achieve a basis we may define $\xi^{(2)} = (0, 1)^t$ and represent arbitrary initial data in the form

$$\xi(0) = c_1 \xi^{(1)} + c_2 \xi^{(2)}.$$

Direct verification shows that

$$\xi(t) = \left\{ (c_1 + c_2 t)\, \xi^{(1)} + c_2 \xi^{(2)} \right\} \exp \lambda t$$

is the solution of equation (8.4) taking on the prescribed initial data. It is clear that if $\lambda \geq 0$ then the origin is unstable. If $\lambda < 0$ then, while the term $t \exp \lambda t$ increases initially, it reaches a finite maximum and $\|\xi(t)\|$ can be made arbitrarily small by choosing c_1 and c_2 sufficiently small; this implies not only stability but also asymptotic stability. □

8.2.2 The case when the eigenvalues are complex

Here $\lambda = \rho \pm i\sigma$ and we may assume that $\sigma \neq 0$ for otherwise the eigenvalue is real (and of multiplicity two), and is discussed above. We could leave the solution in the form given by equation (8.5) above with the proviso that $c_2 = \overline{c_1}$ for real solutions, but it is instructive to adopt an alternative approach.

Let $\xi^{(1)} = \eta + i\zeta$ where η and ζ are the real and imaginary parts of the complex vector $\xi^{(1)}$. Then (verify this!)

$$A\eta = \rho\eta - \sigma\zeta \text{ and } A\zeta = \sigma\eta + \rho\zeta.$$

The general solution of equation (8.4) takes the form $\xi(t) = a(t)\eta + b(t)\zeta$ for real, time-dependent functions a, b provided that they satisfy the equations

$$\dot{a} = \rho a + \sigma b, \quad \dot{b} = -\sigma a + \rho b. \tag{8.6}$$

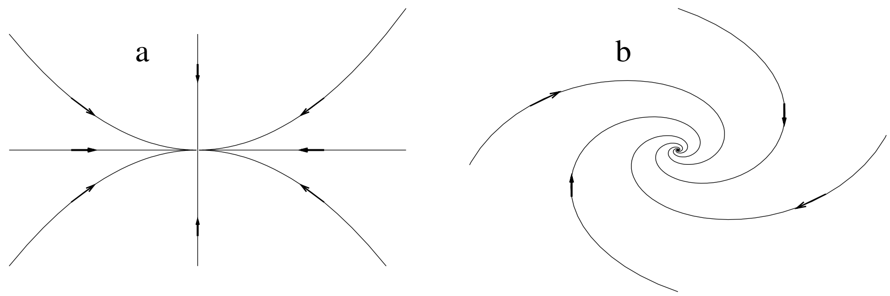

Figure 8.1: Two examples of orbits in the x_1x_2-plane in the case when the origin is a stable equilibrium point of the system (8.4): (a) when both eigenvalues are real (and not equal), and (b) when they are complex conjugates with negative real part.

It is easy to see that these have the general solution

$$a(t) = \exp \rho t \{a_0 \cos \sigma t + b_0 \sin \sigma t\}, \quad b(t) = \exp \rho t \{-a_0 \sin \sigma t + b_0 \cos \sigma t\}. \tag{8.7}$$

This shows that the origin is stable if $\rho \leq 0$ and asymptotically stable if ρ is strictly negative; it is unstable otherwise.

We have arrived, in the present case restricted to $n = 2$, at the general conclusion regarding linear stability (embodied in Theorem 8.3.2 below): if the real part of any eigenvalue is positive we conclude instability and if the real part of each eigenvalue is negative we conclude stability. In the case when one or more eigenvalues has a vanishing real part, either stability or instability is possible and we can draw no conclusion.

Figure 8.1 provides pictures of the orbits in asymptotically stable cases.

8.2.3 Canonical forms for matrices

The pictures of the orbits given in Figure 8.1 can easily be generalized to other cases (see for example [4] for a more complete set of diagrams). However, even for the asymptotically stable cases indicated in that figure, the pictures shown are simplified. This simplified form can be obtained by a change of coordinates that brings the matrix A of the system to one of the two forms

$$(a): \; A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } (b): \; A_2 = \begin{pmatrix} \rho & \sigma \\ -\sigma & \rho \end{pmatrix}. \tag{8.8}$$

These are canonical forms for the matrix A and may be arrived at as follows (cf. [7]).

Consider first the case when the eigenvalues λ_1 and λ_2 of A are both real with linearly independent column eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$. Form the matrix S whose columns are $\xi^{(1)}$ and $\xi^{(2)}$, and let S^{-1} be its inverse (how do we know that this exists?). If in equation (8.4) we make the substitution $x = Sy$, we find that y satisfies the linear, differential equation $\dot{y} = A_1 y$ where A_1 is the matrix given in equation (8.8). The orbits of *this* system are shown in Figure 8.1 (a): to find the orbits of the original system one would transform back to the x system resulting in axes that are in general not at right angles to one another and to corresponding distortions of the pictured orbits.

In the exceptional case alluded to above in which both eigenvalues have the same value λ and there is only one linearly independent eigenvector $\xi^{(1)}$, we may introduce a second vector (sometimes called a *generalized eigenvector*) $\xi^{(2)}$ such that $A\xi^{(2)} = \lambda\xi^{(2)} + \xi^{(1)}$. Introducing the transformation matrix S as above now leads to the system $\dot{y} = A_3 y$ where A_3 is the matrix of Example 8.2.1 above.

Finally, in the case of a complex conjugate pair of eigenvalues we introduce the column vectors η and ζ of §8.2.2 and form the matrix S from these. Transforming again via $x = Sy$ we get the equations (8.6) with $y_1 = a, y_2 = b$. The orbits of these equations are those that are shown in Figure 8.1 (b). To get the pictures in the original x system, we again need to transform back, resulting in non-orthogonal axes and distortions of the pictured orbits.

It is clearly useful to pass to coordinate systems in which the equations take simple forms. These are canonical forms that a matrix may take (the *Jordan* canonical form in the cases of matrices A_1 and A_3, the *real* canonical form in the case of the matrix A_2). Below we describe the Jordan canonical form in generality. Its derivation, which is a generalization of that sketched above in the two-dimensional case, is available in many books treating linear algebra (cf. [7],[9]).

8.3 Linear Stability: general values of n

We now consider the equation (8.4) when the constant-coefficient matrix A is of arbitrary dimension n. We will be interested mostly in the case when the matrix A, the dependent variable x and the time t are real, but most of the development below applies also when they are complex. Even when A

and t are restricted to real values, it will be useful to allow x to take complex values.

In §3.6.2 above, we have defined the matrix exponential function $\exp At$ and shown that it is the fundamental matrix solution of equation (8.4); in other words, the solution of that equation taking the initial value x_0 is $x(t) = (\exp At)\, x_0$. It is therefore clear that the stability or instability of the zero solution of equation (8.4) is entirely determined by the behavior of $\exp At$ as $t \to +\infty$ so we now address this.

The matrix, or operator norm, is defined in general by $\|A\| = \sup_{\|x\|=1} \|Ax\|$, in terms of the vector norm[2]. A choice of norm that we will sometimes make below is

$$\|x\| = \sum_{i=1}^{n} |x_i|. \tag{8.9}$$

One can then show that the matrix norm is given by the formula

$$\|A\| = \sup_j \sum_{i=1}^{n} |A_{ij}|. \tag{8.10}$$

Here the notations $|x_i|$ and $|A_{ij}|$ represent the moduli of the corresponding complex numbers.

A transformation $x = Py$ for any constant, nonsingular matrix P may be viewed as a change of coordinates, from $y_1, y_2, \ldots, y_n$ to $x_1, x_2, \ldots, x_n$. If we introduce this transformation into equation (8.4) we find a similiar equation

$$\dot{y} = By, \quad B = P^{-1}AP. \tag{8.11}$$

Thus the matrix B is *similar* to A. This equation has the fundamental matrix solution $\exp Bt$. Since the relation $P^{-1}AP = B$ implies that $P^{-1}A^kP = B^k$ for any non-negative integer k, it follows from the power-series definition of $\exp At$ that $P^{-1}e^{At}P = e^{Bt}$.

We now choose P to reduce A to Jordan canonical form J. We can write J in the following form involving $r+1$ block matrices $J_k, (k = 0, 1, ..., r)$ along the diagonal:

$$J = \begin{pmatrix} J_0 & 0 & \cdots & 0 \\ 0 & J_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_r \end{pmatrix} \tag{8.12}$$

[2]This is discussed in Chapter 6; see in particular Problem Set 6.2.1, problems 9 and 10.

where J_k is $n_k \times n_k$. Here $J_0 = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_p)$ are eigenvalues (not necessarily distinct) belonging to one-dimensional eigenspaces. To each $k = 1, \dots, r$, there corresponds an eigenvalue λ_{p+k} and a generalized eigenspace of dimension $n_k \geq 2$. The corresponding Jordan block may be written

$$J_k = \lambda_{p+k} I_{n_k} + Z_{n_k}, \tag{8.13}$$

where I_m is the $m \times m$ identity matrix and Z_m is an $m \times m$ nilpotent matrix of the form

$$Z = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8.14}$$

The matrix Z has ones on the secondary diagonal above the main diagonal and all other entries are zero. It is easy to check that $Z^m = 0$ whereas $Z^k \neq 0$ for $k = 1, 2, \dots, m-1$.

It is worth remarking here that the Jordan canonical form (8.12) is not unique, since the ordering of the blocks $J_1, \dots, J_r$ is arbitrary, as is the distribution of the eigenvalues appearing in J_0, since each of the latter belongs to a one-dimensional eigenspace.

Since the block structure of J is preserved under multiplication and addition it follows that

$$e^{Jt} = \begin{pmatrix} e^{J_0 t} & 0 & \cdots & 0 \\ 0 & e^{J_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_r t} \end{pmatrix}. \tag{8.15}$$

Now for any of the matrices J_k, the exponential matrix $e^{J_k t}$ has a simple form. It is particularly simple for J_0 viz.

$$e^{J_0 t} = \text{diag}(e^{\lambda_0 t}, \dots, e^{\lambda_p t}). \tag{8.16}$$

For $k \geq 1$, according to equation (8.13), J_k is the sum of two matrices which commute because one of them is a multiple of the identity. Hence we find (see Proposition 3.6.1 of Chapter 3)

$$e^{J_k t} = e^{(\lambda_{p+k} I + Z_{n_k})t} = e^{\lambda_{p+k} t} e^{Z_{n_k} t}, \tag{8.17}$$

i.e., each block beyond the first is the product of a complex number $e^{\lambda_{p+k} t}$ with the exponential of a nilpotent matrix. The latter is easily expressible in terms of the power series expansion, because that series terminates: if $Z = Z_m$, then $Z^n = 0$ if $n \geq m$. An easy calculation now gives

$$e^{Zt} = \begin{pmatrix} 1 & t & \cdots & t^{m-1}/(m-1)! \\ 0 & 1 & \cdots & t^{m-2}/(m-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{8.18}$$

This gives a complete description of the e^{At} in terms of the eigenvalues of A and the structure of the generalized eigenspaces.

Up to the present, whereas we may have thought of t as real, we have not needed this assumption. In the next theorem, however, we do need to assume that t is real.

Theorem 8.3.1 *If $Re(\lambda_k) \leq \alpha$ for each $k = 1, \dots, p+r$, then there exists a constant C and and an integer $m \geq 0$ such that, $\forall\, t \geq 0$,*

$$\|e^{At}\| \leq C\left(1 + t + \cdots + \frac{t^m}{m!}\right) e^{\alpha t}. \tag{8.19}$$

Proof: First consider $\|e^{Jt}\|$. From (8.15) we see that the matrix e^{Jt} preserves the block structure of J. The norm (8.10) is a supremum over column sums, and by the block structure we see that any column sum includes entries from only a particular block $e^{J_k t}$. Thus it is clear that

$$\|e^{Jt}\| \leq e^{\alpha t}\left[1 + t + \cdots + \frac{t^m}{m!}\right],$$

where the integer m is at most 1 less than the dimension of the largest generalized eigenspace.

Since $P^{-1}e^{At}P = e^{J(t)}$, we have

$$\|e^{At}\| \leq C\|e^{Jt}\| \quad \text{where}\, C \geq \|P\|\|P^{-1}\|. \quad \square$$

Remark: We have referred specifically to the norm (8.10) in the proof of this theorem, but the conclusion is independent of the choice of norm in virtue of the fact that, in finite dimensions, any two norms are equivalent; see §6.2 above.

Corollary 8.3.1 *For any $\beta > \alpha$ and some $K > 0$, $\|e^{At}\| \leq Ke^{\beta t}$.*

Proof: It's only necessary to observe that, for any $\epsilon > 0$, the function $t^k \exp(-\epsilon t)$ has a maximum for positive t. The conclusion follows on putting $\epsilon = \beta - \alpha$. □

This result has the following important special case:

Corollary 8.3.2 *If* $Re(\lambda_k) < 0$ *for each* k*, then* $\|e^{At}\| \leq Ke^{-\gamma t}$ *for some* $\gamma > 0$*.*

The following conclusions for the stability or instability of the linear system (8.4) of size n are immediate:

Theorem 8.3.2 *The origin is asymptotically stable for the system (8.4) if the real part of every eigenvalue is negative. It is unstable if any eigenvalue has a positive real part.*

PROBLEM SET 8.3.1

1. Consider the logistic equation, equation (1.30) of Chapter 1. Show that it has two equilibrium solutions and discuss the linearized stability of each.
2. Consider the two-dimensional, linear system $\dot{x} = Ax$ where
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Is the origin stable or unstable?
3. For the system (7.13) of the preceding chapter, determine the linearized stability of each equilibrium point. You may choose $R = 0.2$
4. Consider the two-dimensional, linear system $\dot{x} = Ax$ where
$$A = \begin{pmatrix} -\delta & 1 \\ 0 & -\delta \end{pmatrix}$$
where δ is a positive number so that the origin is asymptotically stable. Define the *amplification factor* for a solution $x(t)$
$$a = \sup ||x(t)||/||x(0)||,$$
where the supremum is taken over all $t > 0$ and over all initial data $x(0) \neq 0$ and the norm is the Euclidean norm: $||x|| = \sqrt{x_1^2 + x_2^2}$. Show that $a \geq 1/(\delta e)$ where e is the base of the natural logarithm.
5. Suppose that the origin is an unstable equilibrium point of the system (8.1) according to Definition 8.1.1. Prove that there exists a positive number ϵ_0 and sequences of vectors $\{x_k\}_1^\infty$ and positive, real numbers $\{t_k\}_1^\infty$ such that $\|x_k\| \to 0$ and $t_k \to \infty$ as $k \to \infty$ and $\|\phi(t_k, x_k)\| \geq \epsilon_0$ for each $k = 1, 2, \ldots$.

6. Prove the formula (8.10).

7. Prove the formulas (8.16), (8.17) and (8.18).

8. For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

find the fundamental matrix solution $\Phi(t) = \exp(At)$ of the equation $\dot{x} = Ax$.

For the next two problems the matrix A is

$$A = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 3/2 \end{pmatrix}. \tag{8.20}$$

9. Verify that a fundamental matrix solution of equation (8.4)[3] with A given by equation (8.20) is

$$\Phi(t) = \begin{pmatrix} (1-t/2)e^t & (t/2)e^t \\ -(t/2)e^t & (1+t/2)e^t \end{pmatrix}.$$

10. Arrive at the formula of problem 9 by using the fact that $\exp(At) = \Phi(t)$. (Hint: find the similarity matrix P taking A to its Jordan form, evaluate $\exp(Jt)$, then transform back via P^{-1} to find $\exp(At)$.)

11. Prove Theorem 8.3.2. Under what conditions on the matrix A of the system (8.4) is the origin stable but not asymptotically stable?

12. In the Jordan form, the nilpotent matrices of the form (8.14) can be exchanged for similar matrices in which the 'ones' on the secondary diagonal are replaced by arbitrary nonzero numbers, e.g., for $\nu \neq 0$,

$$Z_\nu = \begin{pmatrix} 0 & \nu & 0 & \cdots & 0 \\ 0 & 0 & \nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8.21}$$

Prove this.
(Hint: calculate the effect of acting on a Jordan block $B = \lambda I + Z$ of size m with the similarity transformation $R = \text{diag}\left(\nu^{-m}, \nu^{-m+1}, \ldots, \nu^{-1}\right)$.)

[3] *Verify* means: work out each side of equation (8.4) and check that they are the same.

8.4 Nonlinear Stability

In this section we show that, under certain additional assumptions, linearized stability is decisive for stability, i.e., the nonlinear terms in equation (8.3) do not contribute to the qualitative determination of stability. The theorems of this section, Theorems 8.4.1 and 8.4.2 are therefore of wide importance and application.

The basic equation (8.1) may, with $x = q + \xi$, be rewritten as

$$\dot{\xi} = A\xi + g(\xi) \tag{8.22}$$

as in equation (8.3), where A has the same meaning as in that equation, and

$$g(\xi) = f(q+\xi) - f(q) - A\xi = f(q+\xi) - A\xi \tag{8.23}$$

represents the nonlinear terms that were represented by dots in equation (8.3). Note that $g_\xi(\xi) = f_x(q+\xi) - A\xi$ which vanishes if $\xi = 0$. Therefore

$$g(0) = 0 \text{ and } Dg(0) = 0 \tag{8.24}$$

where we have introduced the alternate notation $Dg = g_\xi$ for the matrix of partial derivatives. We therefore consider the equivalent initial-value problem consisting of equation (8.22) with initial data ξ_0. Since $\xi = 0$ is a solution of equation (8.22) for all t, the choice $\xi_0 = 0$ leads to this (and only this) solution. We therefore consider only $\xi_0 \neq 0$.

We begin by reformulating the integral version of the initial-value problem (as in equation (6.14) above). It is easy to check directly that the integral equation

$$\xi(t) = \exp(At)\xi_0 + \int_0^t \exp\left(A(t-s)\right) g\left(\xi(s)\right)\, ds \tag{8.25}$$

is equivalent to the initial-value problem for the system (8.22). We will have more than one use for the following

Lemma 8.4.1 *Suppose the function $f : \Omega \to R^n$ is C^1 on Ω and that $Df(q) = 0$ for some $q \in \Omega$. Then for any $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that*

$$\|f(y)-f(x)\| \le \epsilon_0\|y-x\| \quad \textit{provided} \quad \|x-q\| < \delta_0 \textit{ and } \|y-q\| < \delta_0. \tag{8.26}$$

Proof: For any C^1 function f the identity

$$f(y) - f(x) = \int_0^1 Df(sx + (1-s)y)\,(x-y)\, ds$$

holds, provided x and y are in a sufficiently small ball with center at q. The identity follows from the observation that the integrand is $(d/ds)\, f(sx + (1-s)y)$. Since $Df(q) = 0$, $\|Df(x)\| < \epsilon_0$ if $\|x - q\| < \delta_0$. Assuming that both x and y satisfy the latter inequality, it follows that so also does $sx + (1-s)y$, and the conclusion follows by taking norms on either side of the last equation. □

Corollary 8.4.1 *With g defined by equation (8.23), we have*

$$\|g(\xi)\| < \epsilon_0 \|\xi\| \text{ if } \|\xi\| < \delta_0. \tag{8.27}$$

Proof: We apply the preceding lemma to g with $y = \xi$ and $x = 0$, noting that $g(0) = 0$. □

Now consider the case when all the eigenvalues of the matrix A have negative real parts. According to Corollary 8.3.2, there are positive constants K and γ such that $\|\exp(At)\| \le Ke^{-\gamma t}$ for all $t \ge 0$. Assume without loss of generality that $K \ge 1$. To prove stability in this case, we need to show that, for any given $\epsilon > 0$, $\|\xi(t)\|$ remains less than ϵ provided that $\|\xi_0\| < \delta$ for some sufficiently small δ. In this, we are free to rechoose ϵ to be smaller than the given value. We shall choose it as follows.

In the preceding estimate for g, choose $\epsilon_0 < \gamma/K$. This determines a positive number δ_0. Choose $\epsilon < \delta_0$. This ensures that as long as $\|\xi(t)\| < \epsilon$, we will have $\|g(\xi(t)\| < \epsilon_0$. Now choose $\delta < \epsilon/K$. We now show that if $\|\xi_0\| < \delta$, we must have $\|\xi(t)\| < \epsilon$ for all positive t. Initially this must be so since $\delta < \epsilon$ so it must hold on some interval to the right of the origin. If it should fail for any positive t, then there is a least value, say T, where it first fails. Then

$$\|\xi(T)\| = \epsilon \text{ whereas } \|\xi(t)\| < \epsilon \text{ on } [0, T). \tag{8.28}$$

Consider equation (8.25) for $t \le T$. Taking norms gives

$$\|\xi(t)\| \le K\delta e^{-\gamma t} + \int_0^t Ke^{-\gamma(t-s)} \epsilon_0 \|\xi(s)\|\, ds$$

where we have exploited the inequality (8.28) so as to be able to use the estimate (8.27). Provisionally define

$$u(t) = \|\xi(t)\| e^{\gamma t}.$$

Then the inequality for $\|\xi(t)\|$ takes the form

$$u(t) \le K\delta + \int_0^t K\epsilon_0 u(s)\, ds.$$

Gronwall's lemma now implies

$$u(t) \leq K\delta \exp\left(K\epsilon_0 t\right)$$

or

$$\|\xi(t)\| \leq K\delta \exp\left(K\epsilon_0 - \gamma\right) t.$$

The exponent is negative by our choice of ϵ_0 so $\|\xi(t)\| < K\delta$ on $[0, T)$ and, in particular, $\|\xi(T)\| \leq K\delta < \epsilon$, contradicitng the assumption that $\|\xi(T)\| = \epsilon$. This proves stability in this case. Indeed, a minor modification of the reasoning shows that the stability is asymptotic and we have

Theorem 8.4.1 *Suppose that in equation (8.1) the function $f \in C^1(\Omega)$ where Ω is a domain in R^n, and suppose that $q \in \Omega$ is an equilibrium point at which all the eigenvalues of the Jacobian matrix f_x have negative real parts. Then q is an asymptotically stable equilibrium point of f.*

We now turn to the case when the real parts of n_1 of the eigenvalues of the matrix $A = f_x(q)$ are positive, $n_1 \geq 1$. This of course implies instability for the linearized problem and we'll show that it likewise implies instability for the full, nonlinear problem. Suppose then that the eigenvalues are divided into two groups, one in which all real parts are positive (say $\lambda_1, \ldots, \lambda_{n_1}$) and a second in which all real parts are negative or zero (say $\lambda_{n_1+1}, \ldots, \lambda_n$), with $n_1 + n_2 = n$. In each of these groups we must allow for multiplicities.

We begin by transforming coordinates so that the matrix A takes a canonical form with two special features. The first is a transformation to a Jordan-canonical matrix $B = P^{-1}AP$ such that the eigenvalues with positive real parts come first; this exploits the arbitrariness of the Jordan form with respect to the ordering of the eigenvalues. This will give B a block structure

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

in which each of the matrices B_1 and B_2 possesses the Jordan canonical structure: eigenvalues appear on the diagonal, and a certain number of constants occur along a secondary diagonal; in B_1 each eigenvalue has a positive real part whereas in B_2 no eigenvalue has a positive real part. The second special feature is that the constants along the secondary diagonal are all equal to ν, a positive parameter that we are free to choose (cf. Problem 8.21 of the preceding problem set).

The transformation of variables $\xi = P\eta$ will in general require that P and η be in C^n, i.e., that their components be complex, although ξ is in R^n.

In the transformation of the nonlinear function $h(\eta) = P^{-1}g(P\eta)$ the values of the function h and of its argument η are in C^n, but h is well defined since the argument of g takes only values in R^n. Lemma 8.4.1 has an immediate extension to h and we shall apply it to h with the same notation for the constants ϵ_0, δ_0 indicated in that lemma.

The strategy is to assume that the origin is stable and deduce a contradiction. Denote by σ a positive lower bound for the real parts of the eigenvalues of B_1. We shall in due course choose ϵ_0 and ν sufficiently small in relation to σ. We may assume that δ in the definition of stability is chosen sufficiently small that the norm of the solution remains less than $\epsilon < \delta_0$, so that the estimate of Corollary 8.27 holds for the function h.

The equation (8.22) takes the form

$$\dot{\eta}_1 = B_1\eta_1 + h_1(\eta), \tag{8.29}$$

$$\dot{\eta}_2 = B_2\eta_2 + h_2(\eta), \tag{8.30}$$

where we have written (η_1, η_2) in place of η, splitting the latter in conformity with the block structure of B. We'll use a norm $R = R_1 + R_2$ where

$$R_1 = \sqrt{\sum_1^{n_1} \overline{\eta_{1i}}\eta_{1i}},$$

where the overbar represents the complex conjugate and n_1 is the dimension of η_1. There is a similar expression for R_2. Then

$$\frac{d}{dt}R_1^2 = \sum_{j=1}^{n_1}\left(\overline{\eta_{1j}}\sum_{l=1}^{n_1} B_{1jl}\eta_{1l} + \text{c.c.}\right) + \sum_{j=1}^{n_1}\left(\overline{\eta_{1j}}h_{1j}(\eta) + \text{c.c.}\right)$$

where c.c. means the complex conjugate of the preceding expression. The first of the two terms above, which arises from the linear terms in the equation, can be separated into two parts, one coming from the diagonal part of the matrix B_1 and the other from the secondary diagonal. That coming from the diagonal term exceeds $2\sigma R_1^2$. That coming from the secondary diagonal is at most $2\nu R_1^2$ in magnitude, and therefore exceeds $-2\nu R_1^2$. The final term, arising from the nonlinear term, is easily estimated with the aid of the Schwartz inequality and is seen to exceed $-2\epsilon_0 R_1 (R_1 + R_2)$, provided that $R_1 + R_2$ remains less than δ_0. This gives

$$\frac{dR_1}{dt} \geq (\sigma - \nu - \epsilon_0)R_1 - \epsilon_0 R_2.$$

In a similar way we deduce for R_2 the inequality

$$\frac{dR_2}{dt} \leq (\nu + \epsilon_0) R_2 + \epsilon_0 R_1.$$

Putting these together gives

$$\frac{d}{dt}(R_1 - R_2) \geq (\sigma - 2\nu - 2\epsilon_0) R_1 - 2\epsilon_0 R_2.$$

The choices $\nu < (3/8)\sigma$ and $\epsilon_0 < (1/8)\sigma$, which we are free to make, now ensures that

$$\frac{d}{dt}(R_1 - R_2) \geq (\sigma/2)(R_1 - R_2).$$

Choosing initial data so that $R_1 > R_2$ at $t = 0$ is now seen to imply that $R_1 - R_2$ increases without bound, contradicting the assumption that $R_1 + R_2 < \epsilon$. This proves

Theorem 8.4.2 *Suppose that in equation (8.1) the function $f \in C^1(\Omega)$ where Ω is a domain in R^n, and suppose that $q \in \Omega$ is an equilibrium point at which at least one eigenvalue of the Jacobian matrix f_x has a positive real part. Then q is an unstable equilibrium point of f.*

The two theorems of this section may be combined into one:

Theorem 8.4.3 *Suppose the C^1 system 8.1 possesses the equilibrium point p and put $T = f_x(p)$. The origin is asymptotically stable if the real part of every eigenvalue of T is negative. It is unstable if any eigenvalue of T has a positive real part.* □

Comparing this with Theorem 8.3.2, we see that its conclusions are decisive also for nonlinear stability.

Theorem 8.4.3 covers a lot of ground and is among the most widely quoted theorems in dynamical-systems theory. However, there are patches of ground that it does not fully cover. We explore an interesting such patch in the next section.

8.5 Conservative Systems

The autonomous system (8.1) will be called conservative if there exists a C^1 scalar function $E : \Omega \to R$ which is not constant on any open set in Ω, but is constant on orbits. The function E is called an *integral*, or a *constant of the motion*, of the system (8.1).

Theorem 8.5.1 *An equilibrium point q of a conservative system cannot be asymptotically stable.*

Proof: Suppose q is an asymptotically stable equilibrium point. Then there is a neighborhood N of q such that if $p \in N$, $\phi(t,p) \to q$ as $t \to \infty$. However, if the system is assumed conservative with integral E, then E is constant on orbits, so $E(p) = E[\phi(t,p)] = E(q)$ for each $p \in N$. This implies that E is constant on an open set, which is a contradiction. □

This shows that proving stability for a conservative system cannot rely on Theorem 8.4.1 and will require different methods from those employed there.

When a conservative system has an equilibrium point at which the integral has a minimum, an inference of stability can be made on this basis. In particular

Definition 8.5.1 *A function $E : \Omega \to R$ is said to have a* strong *minimum at q if there is a neighborhood N of q such that $E(x) > E(q)$ for every $x \in N$ except for $x = q$.*

It is the strictness of the inequality in this definition that merits the adjective *strong*. If one knew only that $E(x) \geq E(q)$ in N then E would have a minimum there but it would not qualify as strong and one could not draw the conclusion of the following theorem.

Theorem 8.5.2 *Suppose q is an equilibrium point of a conservative, autonomous system and that its integral E has a strong minimum there. Then q is stable.*

Proof: We may suppose $q = 0$. Let $V(q) = E(q) - E(0)$. Given $\epsilon > 0$ consider the set $\{q : \|q\| = \epsilon\}$. This lies in the neighborhood N if ϵ is small enough and we may assume this is so. Denote by V_ϵ the minimum of V on the set $\|q\| = \epsilon$. Choose $\delta < \epsilon$ such that $V(q) < V_\epsilon$ if $\|q\| < \delta$. Then $\|q(t)\| < \epsilon$ for all $t > 0$. □

The same conclusion can be drawn if E has a strong maximum at equilibrium.

Hamiltonian dynamics is a formulation of the dynamics of point masses with wide applicability. Let q and p be n-vectors (coordinates and momenta, respectively). Hamilton's canonical equations are:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Here the Hamiltonian function $H = H(q, p)$ is a function of $2n$ variables and the system is said to have n degrees of freedom. It follows easily from the equation that if, as indicated here, H is not explicitly dependent on time, it is constant on orbits. Hence if H is either a maximum or a minimum at equilibrium, that equilibrium is stable.

Example 8.5.1 (Linearized instability need not imply instability) Let $H = p^4 + q^2$. The origin is an equilibrium point. The linearized equations are $dq/dt = 0, dp/dt = -2q$. This has solutions increasing linearly with time, so the origin is unstable for the linearized system. However, the Hamiltonian has a minimum there, so it is stable. □

In Hamiltonian systems, linearized stability of the origin is often characterized by motions of an oscillatory character, and nonlinear stability is sometimes taken for granted if linearized stability holds. This does not follow from Theorem 8.4.1 and is a somewhat risky conclusion, particularly when there can be resonant interactions in the nonlinear system. The following example in two degrees of freedom shows this explicitly.

Example 8.5.2 (Linearized stability need not imply stability)

$$H = \frac{1}{2}\left[q_1^2 + p_1^2\right] - \left[q_2^2 + p_2^2\right] + \frac{1}{2}p_2\left[p_1^2 - q_1^2\right] - q_1 q_2 p_1.$$

Then the equations are

$$\begin{aligned} \dot{q}_1 &= p_1 + p_1 p_2 - q_1 q_2 & \dot{p}_1 &= -q_1 + q_1 p_2 + p_1 q_2 \\ \dot{q}_2 &= -2p_2 + (1/2)(p_1^2 - q_1^2) & \dot{p}_2 &= 2q_2 + q_1 p_1. \end{aligned}$$

The linearized system can be read off this system; all its solutions are bounded, and it possesses periodic solutions with periods 2π and 4π. However, for arbitrary T a solution to the full system is

$$p_1 = \sqrt{2}\frac{\sin(t-T)}{t-T},\ p_2 = \frac{\sin 2(t-T)}{t-T},\ q_1 = \sqrt{2}\frac{\cos(t-T)}{t-T},\ q_2 = \frac{\cos 2(t-T)}{t-T}.$$

For small t these approximate to solutions of the linearized system, but if $T > 0$ they become unbounded in finite time. □

PROBLEM SET 8.5.1

1. The reasoning leading to the conclusion of asymptotic stability in Theorem 8.4.1 requires a "minor modificaton." Provide it.

   The following three problems relate to the Lorenz system (8.2).

2. The origin is an equilibrium solution of the Lorenz system for all values of r. For what values of r is it stable?

3. Find the equilibrium solutions of the Lorenz system *other* than the origin. Show that they exist as real solutions only for $r > 1$.

4. Formulate the linearized-stability problem for the equilibrium solutions of Problem 3 and write out the characteristic polynomial $p(\lambda)$ for the eigenvalues.

5. Find the roots of the characterstic equation $p(\lambda)$ of Problem 4 explicitly if $r = 1$. Show that the nonzero solutions are stable for values of $r > 1$, at least if $r - 1$ is not too large.
   (Hint: put $r = 1 + \epsilon$ and seek roots $\lambda = \lambda_0 + \lambda_1 \epsilon + \ldots$.)

   The next three problems relate to the Lotka-Volterra system (see Example 7.3.3 of Chapter 7).

6. Find the equilibrium solutions of the Lotka-Volterra system and discuss their linearized stability.

7. Show that the Lotka-Volterra system is conservative by obtaining a function $u(x, y)$ that is constant on orbits.
   (Hint: try $u(x, y) = Ax + By + C \ln x + D \ln y$ where A, B, C, D are constants to be determined.)

8. Use the function u of the preceding problem to infer that the nonzero equilibrium point of the Lotka-Volterra system is stable.

   The following three problems relate to the equations of *rigid-body dynamics*, which may be written (cf. Problem 13 of Problem Set 6.2.1)

   $$I_1 \dot{\omega}_1 + (I_2 - I_3)\omega_2 \omega_3 = 0, \; I_2 \dot{\omega}_2 + (I_3 - I_1)\omega_3 \omega_1 = 0, \; I_3 \dot{\omega}_3 + (I_1 - I_2)\omega_1 \omega_2 = 0.$$

   Here $\omega_1, \omega_2, \omega_3$ are the components of the angular-velocity vector along the principal axes of the rigid body, and I_1, I_2, I_3 are positive constants: the moments of inertia about the corresponding axes.

9. Show that, aside from the trivial equilibrium solution $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$, there are three families of equilibrium solutions of the form $(\omega_1, 0, 0), (0, \omega_2, 0), (0, 0, \omega_3)$.

10. Verify that the equilibrium solution $(\omega_1, 0, 0)$ is (linearly) stable if the corresponding moment of inertia I_1 is either the greatest or the least of the three moments of inertia, but unstable if it is intermediate between I_2 and I_3.

11. Find *two* constants of the motion for this system.

12. Refer to Lemma 8.4.1 and suppose ϵ_0 is chosen to lie in the interval $(0,1)$. Show that f has a *fixed point* p, i.e., a point p such that $f(p) = p$, with $\|p - q\| < \delta_0$. (This is a version of the *contraction-mapping theorem.*)

8.6 Invariant Sets and Manifolds

We have encountered invariant sets of a special kind in Chapter 7. We now consider further such sets that are of great use in attaining a qualitative understanding of dynamical systems. An invariant set need not have the structure of a manifold but we next consider some that do indeed have that structure.

8.6.1 The Stable Manifold

Consider the system (8.1) rewritten as

$$\dot{z} = f(z) \tag{8.31}$$

where we have written z in place of x since we use x for a different vector for the purposes of this section. We suppose the system (8.31) has an equilibrium point at the origin. If all the eigenvalues of the linearization $A = Df(0) = f_z(0)$ have negative real parts then the origin is stable. We wish to consider the case when some but not all of the eigenvalues have negative real parts. It is then the case that the linearized system (8.4) possesses a stable subspace, i.e., a subspace E_s of R^n with the property that, if $\xi_0 \in E_s$, then the solution $\xi(t)$ of the linearized system tends to the origin as $t \to +\infty$. We may see this with the aid of the following

Theorem 8.6.1 *Let the real, $n \times n$ matrix A have k eigenvalues with negative real parts. Then there is a real transformation matrix S such that*

$$S^{-1}AS = \begin{pmatrix} C & O \\ O & D \end{pmatrix} \tag{8.32}$$

where C is a $k \times k$ matrix all of whose eigenvalues have negative real parts, D is an $(n-k) \times (n-k)$ matrix all of whose eigenvalues have real parts that are positive or zero, and matrices marked O are $k \times (n-k)$ and $(n-k) \times k$ matrices of zeros.

Example 8.6.1 In the linear, two-dimensional system $\dot{\xi} = A\xi$ let

$$A = \begin{pmatrix} 5/3 & 4/3 \\ -4/3 & -5/3 \end{pmatrix}.$$

One finds easily that the eigenvalues are $\lambda = \pm 1$ with the corresponding eigenvectors

$$\xi_+ = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi_- = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The transformation matrix S is formed from these columns and, with the transformation $y = Sx$ we find that the original system may be expressed as

$$\dot{y}_1 = y_1, \quad \dot{y}_2 = -y_2.$$

The stable subspace is now $y_1 = 0$. In terms of the original variables, for which $y_1 = 2x_1 + x_2$, it is $2x_1 + x_2 = 0$. $\square$

We will not give a formal proof of Theorem 8.6.1 but make the following observations about it. If all eigenvalues are real, the Jordan canonical form effects such a transformation. If some eigenvalues are complex, say $\lambda = \rho + i\sigma$, then the corresponding eigenvector $\xi = \eta + i\zeta$ is also complex, and these come together with their complex conjugates. Then using the real vectors η, ζ as columns in the transformation matrix will result, not in diagonal entries in the transformed matrix, but in a 2×2 block along the principal diagonal. Similar remarks apply to generalized eigenvectors and lead to blocks along the principal diagonal as well as along secondary diagonals. This leads to the so-called *real canonical form* for the real matrix A. For more details regarding this, see, for example the book [7].

In the example (8.6.1) above, the stable, linear subspace is invariant: if $y_1 = 0$ initially, it remains zero for all time. The same is true in a general, linear problem $\dot{\xi} = A\xi$. Transform this system as in Theorem (8.6.1) and decompose the n-component vector ξ into (x, y), subvectors of k and $n - k$ components respectively, so that the system (8.4) is written $\dot{x} = Cx, \dot{y} = Dy$. The stable, linear subspace is obtained by choosing $y(0) = 0$, which leaves $y(t) = 0$ for all t.

The stable manifold theorem below asserts the existence of an invariant manifold M_s of solutions having the property that, for x_0 on that manifold, the solution $\phi(t, x_0) \to 0$ as $t \to \infty$. Moreover, the manifold M_s will be seen to be tangent to the stable subspace E_s of the linearized system at the origin of coordinates.

In the remainder of this section we suppose that the transformation from z to $\xi = (x, y)$ has been effected (i.e., $z = S\xi$), and further analysis will be carried out in this coordinate system. It should be clear that the stable manifold of the linear problem, when described in the z system, takes the form

$$z_i = \sum_{j=1}^{k} S_{ij} x_j, \ i = 1, 2, \dots n$$

for $x \in R^k$. Similarly, when we find below a local manifold with description $y = h(x)$ in the new (ξ) coordinate system, this corresponds in the original (z) coordinate system to the description

$$z_i = \chi(x) = \sum_{j=1}^{k} S_{ij} x_j + \sum_{j=k+1}^{n} S_{ij} h_j(x), \ i = 1, 2, \dots n,$$

where x now lies in a neighborhood of the origin in R^k.

Example 8.6.2 Consider the system

$$\dot{y}_1 = y_1 + y_2^2, \quad \dot{y}_2 = -y_2.$$

It is easily solved exactly to give

$$y_1 = (a_1 + a_2^2) e^t - a_2^2 e^{-2t}, \quad y_2 = a_2 e^{-t}.$$

Here $a = y(0)$. If $a_1 = -a_2^2$ it is seen that solutions decay to zero as $t \to \infty$. Furthermore, in this case, $y_1(t) = -y_2(t)^2$. The formula $y_1 = -y_2^2$ thus defines the stable manifold in this case. □

Returning to the general case (8.31) and making the assumptions described above – $f(0) = 0, A = f_z(0)$ – we may rewrite this system as $\dot{\xi} = A\xi + g(\xi)$, as in equation (8.22) above. The function g obeys the same estimate as that given in Lemma 8.4.1. We then apply the transformation taking A into the block-diagonal form to get the system

$$\dot{x} = Cx + h(x, y), \quad \dot{y} = Dy + k(x, y), \tag{8.33}$$

where h and k satisfy the estimate of Corollary 8.4.1 (this is similar to the reduction of equation (8.22) above except that there we allowed the transformed variables to be complex). The initial-value problem for this system is equivalent to the integral system

$$x(t) = e^{Ct} x_0 + \int_0^t e^{C(t-s)} h(x(s), y(s)) \, ds, \tag{8.34}$$

$$y(t) = e^{Dt} y_0 + \int_0^t e^{D(t-s)} k(x(s), y(s)) \, ds. \tag{8.35}$$

However, we are concerned now not with the general initial-value problem for arbitrary y_0. We want instead to find $y_0 = \psi(x_0)$ so that solutions with these initial data tend to zero as $t \to \infty$. Rewriting equation (8.35) in the form

$$y(t, x_0, y_0) = e^{Dt} \left(y_0 + \int_0^t e^{-Ds} k\left(x(s, x_0, y_0), y(s, x_0, y_0)\right) ds \right)$$

we see that the vanishing for large t is impossible unless

$$y_0 = -\int_0^\infty e^{-Ds} k\left(x(s, x_0, y_0), y(s, x_0, y_0)\right) ds, \tag{8.36}$$

and this must therefore implicitly define the stable manifold – if there is one! Using it to replace y_0 in equation (8.35), we see that the system we need to solve is

$$x(t, x_0, y_0) = e^{Ct} x_0 + \int_0^t e^{C(t-s)} h\left(x(s, x_0, y_0), y(s, x_0, y_0)\right) ds \tag{8.37}$$

$$y(t, x_0, y_0) = -\int_t^\infty e^{D(t-s)} k\left(x(s, x_0, y_0), y(s, x_0, y_0)\right) ds. \tag{8.38}$$

If we succeed in solving the system (8.37, 8.38) then equation (8.36) establishes a relation

$$y_0 = \varphi(x_0, y_0) \tag{8.39}$$

where φ is shorthand for the integral appearing on the right-hand side of equation (8.36). The invariance indeed follows: on the right-hand side of equation (8.38) replace s with $t + u$:

$$y(t, x_0, y_0) = -\int_0^\infty e^{-Du} k\left(x(t + u, x_0, y_0), y(t + u, x_0, y_0)\right) du$$

and, in the integrand, $x(t+u, x_0, y_0) = x(u, x(t, x_0, y_0), y(t, x_0, y_0))$ and likewise for y, by virtue of the dynamical-systems properties of solutions. Comparison with equation (8.36) then shows that $y(t, x_0, y_0) = \varphi(x(t, x_0, y_0), y(t, x_0, y_0))$.

As in the theorems of stability and instability, we require estimates on the exponential factors. In equation (8.37) the matrix C is multiplied by a positive number (t or $t - s$ with $t > s$) and the estimate of Corollary (8.3.2) gives

$$\|e^{Cv}\| \leq K e^{-\gamma v}, v > 0. \tag{8.40}$$

Here $\gamma > 0$ and may be chosen to be any positive number smaller than the least of the positive numbers $\{\text{Re}(-\lambda_j)\}$, where the $\{\lambda_j\}$ are the eigenvalues of C. The number K is "sufficiently large," as in the proof of the Corollary, and we shall always assume $K \geq 1$. In equation (8.38) the matrix D in the exponent is multiplied by a negative number. The fact that the eigenvalues $\{\mu_j\}$ of D have non-negative real parts then provides a similar estimate in that case:

$$\|e^{Dv}\| \leq Ke^{-\rho v}, v < 0. \tag{8.41}$$

Here $\rho > 0$ is any positive number. This allows us to include the possibility that some of the eigenvalues μ_j have real parts equal to zero. The estimate (8.41) grows exponentially with decreasing, negative v. We choose ρ so that

$$\gamma - 2\rho > 0 \tag{8.42}$$

where γ appears in the estimate (8.40).

We also need an estimate of the nonlinear term

$$g(\xi) = g(x, y) = (h(x, y), k(x, y)).$$

This is provided by Lemma 8.4.1 above, where we have g in place of f, ξ in place of x and 0 in place of q. To be explicit, given any $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that

$$\|g(\xi) - g(\xi')\| < \epsilon_0 \|\xi - \xi'| \text{ if } \|\xi - \xi'\| < \delta_0. \tag{8.43}$$

For ease of application in the estimates below, we may assume that norms have been chosen in the ξ, x and y spaces such that $\|\xi\| = \|x\| + \|y\|$; there is no loss of generality in this.

Theorem 8.6.2 Stable-Manifold Theorem *Let the eigenvalues of C have negative real parts and those of D have non-negative real parts. Then there is a neighborhood N of the origin in R^k and a C^1 function $\psi : N \to R^{n-k}$ vanishing at the origin together with its first derivatives and such that the manifold $y = \psi(x)$ is invariant under the flow of the differential system (8.33); on this manifold all orbits tend to the origin as $t \to 0$.*

Proof: The manifold is implicitly defined by equation (8.36), which can only be evaluated once the system defined by equations (8.37) and (8.38) is solved. We construct the solution to this system by successive approximations, beginning with

$$\left(x^{(0)}(t), y^{(0)}(t)\right) = (0, 0).$$

Subsequent approximations are therefore given by

$$x^{(k+1)}(t,x_0,y_0) = e^{Ct}x_0 + \int_0^t e^{C(t-s)}h\left(x^{(k)}(s,x_0,y_0), y^{(k)}(s,x_0,y_0)\right) ds \quad (8.44)$$

$$y^{(k+1)}(t,x_0,y_0) = -\int_t^\infty e^{D(t-s)}k\left(x^{(k)}(s,x_0,y_0), y^{(k)}(s,x_0,y_0)\right) ds \quad (8.45)$$

Note in particular that $\left(x^{(1)}(t), y^{(1)}(t)\right) = \left(e^{Ct}x_0, 0\right)$. These successive functions must of course lie in the domain where the vector field $g = (h,k)$ is defined. This will be so if they are confined to a sufficiently small ball, say $\|\xi\| < \delta_1$. In view of the structure of the first approximation above, we shall need

$$\|\xi^{(1)}(t)\| = \|e^{Ct}x_0\| \leq Ke^{-\gamma t}\|x_0\|, \quad (8.46)$$

so if we choose

$$\|\xi_0\| = \|x_0\| < \delta_1/2K \quad (8.47)$$

this will ensure that the lowest approximations $\xi^{(0)}$ and $\xi^{(1)}$ lie in the ball of radius δ_1. We may of course choose x_0 smaller, say less than $\delta_0 < \delta_1/2K$, and we shall indeed do this so that not only is the inequality (8.47) satisfied but so also the inequality (8.43) with ϵ_0 chosen so that

$$\epsilon_0 K\left(\frac{1}{\gamma-\gamma'} + \frac{1}{\gamma-2\rho}\right) \leq \frac{1}{2}. \quad (8.48)$$

We now claim that, for each $k = 0,1,2,\ldots$,

$$\|\xi^{(k+1)}(t) - \xi^{(k)}(t)\| \leq \frac{1}{2^k}K\|x_0\|e^{-\gamma' t}, \; 0 < \gamma' < \gamma \quad (8.49)$$

and that $\xi^{(k)}(t)$ lies in the ball of radius δ_1, for all $t \geq 0$. These are true for $k = 0$ (inequality 8.46). Assume that they are true for $j = 0,1,\ldots,k-1$. Then $\xi^{(k)}$ lies in the ball of radius δ_1 since

$$\begin{aligned}\|\xi^{(k)}\| &\leq \|\xi^{(0)}\| + \|\xi^{(1)} - \xi^{(0)}\| + \cdots + \|\xi^{(k)} - \xi^{(k-1)}\| \\ &\leq 0 + K\|x_0\|e^{-\gamma' t}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\right) \leq 2K\|x_0\|.\end{aligned}$$

Therefore $\xi^{(k+1)}$ can be defined.

To verify the validity of the inequality (8.49) we consider first equation (8.44) and note that

$$\begin{aligned}\|x^{(k+1)}(t) - x^{(k)}(t)\| &\leq \int_0^t Ke^{-\gamma(t-s)}\frac{\epsilon_0}{2^{k-1}}\|x_0\|Ke^{-\gamma' s}\, ds \\ &= \frac{K^2\epsilon_0\|x_0\|}{2^{k-1}}e^{-\gamma t}\frac{e^{(\gamma-\gamma')t}-1}{\gamma-\gamma'} \leq \frac{\epsilon_0 K^2\|x_0\|}{2^{k-1}(\gamma-\gamma')}e^{-\gamma' t}.\end{aligned}$$

Next we consider equation (8.45) and find

$$\|y^{(k+1)}(t) - y^{(k)}(t)\| \leq \int_0^t Ke^{-\rho(t-s)}\epsilon_0\|x_0\|Ke^{-\gamma' s}\frac{1}{2^{k-1}}\,ds$$
$$= \frac{K^2\epsilon_0\|x_0\|}{2^{k-1}}\frac{e^{-\rho t}-1}{\gamma'-\rho} \leq \frac{\epsilon_0 K^2\|x_0\|}{2^{k-1}(\gamma'-\rho)}e^{-\gamma' t}.$$

Putting these together using the relation

$$\|\xi^{(k+1)}(t) - \xi^{(k)}(t)\| = \|x^{(k+1)}(t) - x^{(k)}(t)\| + \|y^{(k+1)}(t) - y^{(k)}(t)\|$$

and noting the definition (8.48), we see that the induction hypothesis is verified.

The sequence $\xi^{(k)}$ satisfies (for $n > m$)

$$\|\xi^{(n)} - \xi^{(m)}\| \leq \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{m-1}}\right)K\|x_0\|e^{-\gamma' t} < \left(\frac{1}{2}\right)^{m-2} K\|x_0\|e^{-\gamma\prime t}.$$

It is a Cauchy sequence and therefore convergent. The convergence is uniform as to t for all $t > 0$, so $\xi(t)$ is a continuous function of t. For $\|x_0\| < \delta_0 \leq \delta_1/2K$, the convergence is likewise uniform as to x_0. The successive approximations are continuous in x_0 and therefore so is the solution.

Return now to equation (8.39). With the solution $(x(s, x_0, y_0), y(s, x_0, y_0))$ now defined for all $s \geq 0$, this takes the form of equation (8.39) where we recall that φ represents the integral on the right-hand side of equation (8.36). We seek to solve this in the form $y_0 = \psi(x_0)$. We approach this by defining

$$F(x_0, y_0) = y_0 - \varphi(x_0, y_0)$$

and seeking a solution of the equation $F(x_0, y_0) = 0$ via the implicit-function theorem. We know that $F(0, 0) = 0$, because if $x_0 = 0$ and $y_0 = 0$ the unique solution for $x(t), y(t)$ is $0, 0$. Then $\varphi = 0$ because the function k vanishes if $(x(t), y(t) = (0, 0)$. Next we consider the Jacobian matrix of F with respect to y_0:

$$\frac{\partial F}{\partial y_0} = I_{n-k} - \frac{\partial \varphi}{\partial y_0}.$$

Formally differentiating the expression for φ gives

$$\frac{\partial \varphi}{\partial y_0} = \int_0^\infty e^{-Ds}\left(\frac{\partial k}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial k}{\partial y}\frac{\partial y}{\partial y_0}\right)ds. \tag{8.50}$$

Putting aside for the moment the question of the legitimacy of differentiating under the integral sign we see that if $(x_0, y_0) = (0, 0)$ then $\partial\phi/\partial y_0 = 0$, because with those initial data $(x(t), y(t)) = (0, 0)$ and the partial derivatives of k vanish at the origin. This shows that the Jacobian matrix of F with respect to y_0 reduces to the identity at the origin, and there is therefore a (unique!) solution $y_0 = \psi(x_0)$ in a neighborhood of the origin. The function ψ vanishes at the origin. As to its derivative $\partial\psi/\partial x_0$, we evaluate it by implicit differentiation. We have

$$0 = \frac{\partial\psi}{\partial x_0} - \frac{\partial\phi}{\partial x_0} - \frac{\partial\phi}{\partial y_0}\frac{\partial\psi}{\partial x_0}.$$

Equation (8.50) shows that $\partial\phi/\partial y_0$ vanishes at $(x_0, y_0) = (0, 0)$ and a similar calculation shows that the same is true of $\partial\phi/\partial x_0$. It follows that $\partial\psi/\partial x_0$ vanishes at the origin.

This completes the proof except for the matter of the legitimacy of differentiating under the integral sign. This would follow from the classical theorem requiring only that the differentiated quantities be uniformly continuous with respect to the integration variable s (they are), if the interval of integration were finite. In the present case this theorem needs to be revisited, and we outline this below (see the argument leading to equation (8.52) below). We leave to the reader the details of applying this reasoning to the present case. □

This theorem is local: we may have found an extremely small part of the stable manifold, or it may in fact be small. When the former is true, it can be extended globally. Denoting by L_s the local stable manifold provided by the theorem, we define (reverting to the notation ϕ for the flow of the system (8.1)

$$M_s = \{y | y = \phi(t, y_0), t < 0, y_0 \in L_s\}. \tag{8.51}$$

Thus we choose points on the locally determined stable manifold and integrate them backwards in time, to $-\infty$ or on a left-maximal interval if that is not possible.

We address here the legitimacy of differentiating under the integral sign in the equation (8.50) above. Consider a scalar function $u(s)$ defined for real values of s in some interval I by means of the formula

$$u(s) = \int_0^\infty v(s, t)\, dt$$

where v and its partial derivative v_s are defined and continuous for $s \in I$ and $t \geq 0$, and the infinite integral converges for each, uniformly as to s. In the case of v_s what this means is that, given $\epsilon > 0$, there is $T > 0$ and independent of s such that

$$|\int_T^\infty v_s(s,t)\,dt| < \epsilon.$$

The candidate for $u'(s)$ is the integral

$$\int_0^\infty \frac{\partial v}{\partial s}(s,t)\,dt.$$

To check this we form the different quotient for u' and subtract this integral:

$$\Delta(h) = \frac{u(s+h)-u(s)}{h} - \int_0^\infty \frac{\partial v}{\partial s}(s,t)\,dt$$
$$\int_0^\infty \left(\frac{v(s+h,t)-v(s,t)}{h} - \frac{\partial v}{\partial s}(s,t)\right)\,dt = \int_0^\infty \left(v_s(s+\theta h,t) - v_s(s,t)\right)\,dt$$

where θ lies in the interval $(0,1)$. We need to show that, given $\epsilon > 0$ we can find $\delta > 0$ such that $|h| < \delta$ implies that $|\Delta(h)| < \epsilon$. First choose $T > 0$ such that

$$\left|\int_T^\infty v_s(s,t)\,dt\right| < \epsilon/4.$$

Then

$$|\Delta(h)| \leq |\int_0^T \left(v_s(s+\theta h,t) - v_s(s,t)\right)\,dt| + \epsilon/2.$$

For s in a compact subset of the interval I and t in the finite interval $[0,T]$ the continuity of $v_s(s,t)$ is uniform and we may find $\delta > 0$ such that $|h| < \delta$ implies that

$$|\left(v_s(s+\theta h,t) - v_s(s,t)\right)| < \epsilon/2T.$$

This shows that $|\Delta(h)| < \epsilon$, i.e., the limit exists, and therefore that

$$u'(s) = \int_0^\infty v_s(s,t)\,dt. \tag{8.52}$$

8.6.2 The basin of attraction

Again suppose the system (8.1) has an equilibrium point at the origin of coordinates O, and that all the eigenvalues of its linearization matrix $Df(0)$ have negative real parts. According to Theorem 8.4.3 the origin is then an asymptotically stable equilibrium point, and for any initial point y that is in a sufficiently small neighborhood of 0, $\phi(t,y) \to 0$ as $t \to +\infty$. However, there may be further initial points y having this property. We consider the maximal such set in this section.

Definition 8.6.1 *Suppose that for the system (8.1) the origin O is an asympotically stable equilibrium point. Its* basin of attraction *B is the set of initial data y such that $\phi(t, y) \to 0$ as $t \to +\infty$.*

It is clear that B is not empty since there is some neighborhood of O that belongs to it. It could be all of R^n – for example, if the equation (8.1) is linear. The following is true in general:

Theorem 8.6.3 *Suppose the dynamical system (7.1) is defined on a domain $D \in R^n$, has a unique solution for each $y \in D$, and that solution is a continuous function of its initial data (cf. Theorem 6.3.1). Let $x = 0$ be an asymptotically stable equilibrium point for this system. Then its basin of attraction is an open, invariant subset of D.*

Proof: There is a neighborhood $N(0)$ of the origin such that, if $q \in N(0)$, then $\phi(t, q) \to 0$ as $t \to \infty$. Suppose now that $y \in B$. Then for some $t_* > 0$ $\phi(y, t_*) = y_* \in N(0)$. By continuity with respect to initial data, there is a neighborhood $N'(y)$ of y such that, if $y' \in N'(y)$, then $\phi(y', t_*) \in N(0)$, and therefore $y' \in B$. This shows that B is open. To see that it is invariant we only need observe that, if $y \in B$ and $y_1 = \phi(t_1, y)$, then the identity $\phi(t, y_1) = \phi(t + t_1, y)$ shows that $y_1 \in B$ as well. □
Remark: any orbit in B exists on $(a, +\infty)$, but it is possible for a to be a finite, negative number, so the invariance of B needs to be understood in the sense that orbits remain in it on their maximal intervals of existence, as discussed in §7.3. The basin of attraction might be all of the domain D,

that is, it is possible that for any $y \in D$ $\phi(t, y) \to 0$ as $t \to \infty$, as simple examples show. However, if B is a proper subset of D, then the open set B has a boundary ∂B. The latter is likewise invariant:

Theorem 8.6.4 *The boundary ∂B of the basin of attraction is an invariant set.*

Proof: Let y_0 be in ∂B and suppose that, for some t_1, $y_1 = \phi(y_0, t_1) \notin \partial B$. It is not possible for y_1 to lie in B, since the latter is invariant by the preceding theorem, whereas y_0 by assumption does not lie in B. Therefore $y_1 \in C$ where C is the complement of $\overline{B}$, the closure of B. Since C is an open set, there is a neighborhood N_1 of y_1 lying entirely in C. By continuity with respect to initial data, there is a neighborhood N_0 of y_0 such that if $y \in N_0$ then $\phi(y, t_1) \in N_1 \subset C$. But, since $y_0 \in \partial B$, any neighborhood of y_0 contains points of B. Choosing for $y \in N_0$ a point of B, and recalling the invariance of B, we arrive at a contradiction. □

8.6.3 The Unstable and Center Manifolds

The unstable manifold of the equilibrium point O is defined as the set of initial data whose orbits tend to O as $to - \infty$. That it indeed is an invariant manifold can be deduced directly from the previous considerations regarding the stable manifold, as follows.

In the system (8.33) above we replace the previous assumption on the decomposition $z = (x, y)$ by the assumption that C is the block of the block-diagonal matrix A all of whose eigenvalues have positive real parts, whereas the block D contains those eigenvalues whose real parts are negative or zero. Then we replace t by $-t$. This reduces the problem of the existence of the unstable manifold to that already proved in Theorem 8.6.2.

There is a related but different invariant manifold associated with eigenvalues with zero real parts, called the center manifold. To discuss the stable manifold we have divided up the eigenvalues into those with negative real parts on the one hand, and those with positive or zero real parts on the other. Now we wish to be more explicit and decompose the vector z into three: $z = (x, v, y)$ with the corresponding decomposition of the system as follows:

$$\dot{x} = Cx + F(x, v, y),\ \dot{v} = Ev + H(x, v, y),\ \dot{y} = Dy + G(x, v, y).$$

The eigenvalues of C all have positive real parts, those of D negative real parts and those of E zero real parts. The dimensions of C, E, D will be k, l, m respectively, where $k + l + m = n$. The l vectors associated with the eigenvalues with zero real part are said to span the center subspace. The center manifold is a manifold invariant under the system of equations above and tangent at the origin to the center subspace. The proof of its existence is omitted (see [3]).

PROBLEM SET 8.6.1

1. Consider the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1,\ \dot{x}_3 = -x_3 + x_1^2 + x_2^2 \tag{8.53}$$

and find the general solution for arbitrary initial data (c_1, c_2, c_3). Find the stable manifold for this system. Also, find a *center manifold*, i.e., a manifold of solutions that is invariant and is tangent to the flat manifold of the linear problem spanned by the vectors belonging to eigenvalues with zero real part.

2. For the system
$$\dot{x} = -x, \ \dot{y} = y + x^2 e^{-t},$$
find the general solution and find a function ψ such that, if $y_0 = \psi(x_0)$, solutions tend to zero as t tends to ∞. Show, however, that this does *not* define an invariant manifold, i.e., it does not imply that $y(t) = \psi(x(t))$.

   On the other hand, find a function $\psi(x,t)$ such that $y_0 = \psi(x_0, 0)$ implies $y(t) = \psi(x(t), t)$.

3. Consider the system
$$\dot{x} = -x(1 + x^2 - 2xy), \ \ \dot{y} = y(1 - xy). \tag{8.54}$$
It has a stable equilibrium point at $(x,y) = (1,1)$ (and another at $(-1,-1)$) and an unstable equilibrium point at the origin. Describe the basin of attraction B of the point $(1,1)$ and relate its boundary ∂B to the stable and unstable manifolds of the point $(0,0)$.

4. Consider the two-dimensional system
$$\dot{x} = -x - x^3, \quad \dot{y} = -y + y^2.$$
Find the equilibrium points and check their stability. Solve explicitly to locate the basin of attraction B of the stable equilibrium point. Show that the maximal interval of existence of any solution starting in B is (a, ∞) where a is a (finite) negative number.

5. Refer to the system (8.33) of dimension n, where x is of dimension k and y of dimension $n-k$. Show that if there is a smooth, invariant manifold of the form $y = F(x)$, the function F must satisfy the *homological equation*
$$DF(x) + k\,(x, F(x)) = F_x(x)\,(Cx + h(x, F(x)))$$
where F_x is the matrix of partial derivatives
$$(\partial F_i/\partial x_j)\,, \ i = 1, \ldots (n-k), \ j = 1, \ldots, k. \tag{8.55}$$

6. Verify that the expressions found in problem (1) for the stable and center manifolds satisfy the homological equation (8.55) above.

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