Chapter 9

Stability II: maps and periodic orbits

Next in simplicity to equilibrium points of the autonomous system $\dot{x} = f(x)$ are periodic orbits. We consider them and their stability in this chapter. For this purpose it is useful to consider *discrete* dynamical systems: mappings obeying conditions (7.6) only for discrete values of t. We begin with these.

9.1 Discrete Dynamical Systems

Let V be a normed vector space over the real numbers, Ω an open set in V and $g : \Omega \to \Omega$ a C^1 mapping. We think of this mapping as a discrete dynamical system in the following sense. Define $g^1 \equiv g$ and $g^n \equiv g \circ g^{n-1}$, i.e., $g^n(x) = g(g^{n-1}(x))$. Then the iterate $x' = g^n(x)$ is the position after n units of time of the point that was initially at position x. The mapping $\phi(n, x) \equiv g^n(x)$ satisfies the conditions for a dynamical system on the set consisting of the non-negative integers. If g is a diffeomorphism, i.e. if g^{-1} exists as a C^1 map from Ω to Ω, then the system is defined for all integers, positive and negative.

Example 9.1.1 Let $g(p) = \phi(T, p)$ where ϕ is the solution of the initial-value problem

$$\dot{x} = f(x), \ x(0) = p \tag{9.1}$$

and T is a fixed, positive number. Suppose further that the domain Ω of ϕ is positively invariant. Then g defines a discrete dynamical system on Ω, by virtue of the second of equations (7.6). If Ω is invariant, i.e , if all solutions starting in Ω remain there for all t positive and negative, then the system is defined for all values of n, since g is a diffeomorphism: $g^{-1}(p) = \phi(-T, p)$. □

A *fixed point* p of g satisfies $g(p) = p$ and therefore $g^n(p) = p$ for all positive integers n. For example, if p is an equilibrium point of the problem (9.1) then it is a fixed point of the solution map ϕ. Stability and instability are defined for such a point as follows:

Definition 9.1.1 *The fixed point p of the mapping g is said to be* stable *if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|g^n(x) - p\| < \epsilon$ for all integers $n > 0$ and for all x such that $\|x - p\| < \delta$. If δ can be chosen not only so that the solution p is stable but also so that $g^n(x) \to p$ as $t \to \infty$, then p is said to be* asymptotically *stable. If p is not stable it is said to be* unstable.

We may again hope to characterize the stability of a fixed point p via the stability of the origin for the linearization of g at p. Putting $x = p + \zeta$ and writing $g(x) = g(p) + T\zeta + \ldots$, we then need to consider the action of the linear map $T : V \to V$. Here $T = g_x(p)$ is the Jacobian matrix of g evaluated at p. Below we shall take the fixed point p to lie at the origin, since this can always be achieved by the shift $x \to p + x$.

9.1.1 Stability of Linear Maps

If $T : V \to V$ is a linear map, then the corresponding dynamical system is

$$x_{n+1} = Tx_n, \quad n = 0, 1, \ldots \tag{9.2}$$

with solution $x_n = T^n x_0$.

Example 9.1.2 Consider the real, one-dimensional, linear map $g(x) = \lambda x$ where λ is a constant. The origin is a fixed point. Since $g^n(x) = \lambda^n x$, it is clear that the origin is asymptotically stable if $-1 < \lambda < 1$, stable but not asymptotically stable if $\lambda = \pm 1$, and unstable if $|\lambda| > 1$. □

The following lemma characterizes the case when the origin is asymptotically stable:

Lemma 9.1.1 *Let $T : V \to V$ be a linear transformation. Then the following are equivalent:*

1. *$|\lambda(T)| < 1$ for each eigenvalue λ of T.*

2. *$T^n x \to 0$ as $n \to \infty \quad \forall x \in V$,*

3. *There is a norm on V such that $\|T\| < 1$.*

Proof: We first show that (2) $\Rightarrow$ (1). Assume (2) holds and suppose there is an eigenvalue λ with $|\lambda| \geq 1$. Let ζ be the eigenvector associated with λ. By (2) we have $\|T^n \zeta\| \to 0$, whereas by our supposition

$$\|T^n z\| \;=\; \|\lambda^n z\| \;\geq\; \|z\| \neq 0,$$

which appears to be a contradiction. It is certainly a contradiction if the eigenvalue λ is real, since then the eigenvector ζ can likewise be chosen to be real. To see that it remains a contradiction if λ is complex, let $\lambda = \alpha + i\beta$ where the imaginary part $\beta \neq 0$, and split $\zeta = \xi + i\eta$ into real and imaginary parts. Since T is a real operator, for any $k = 1, 2, \ldots$,

$$T^k \zeta = T^k \xi + i T^k \eta = \lambda^k \zeta$$

where the middle term gives the real and imaginary parts. Let's choose the Euclidean norm $\|\cdot\|_2$, i.e.,

$$\|\zeta\|_2^2 = \sum_{j=1}^{n} \overline{\zeta_j}\zeta_j = \sum_{j=1}^{n} \left(\xi_j^2 + \eta_j^2\right) = \|\xi\|_2^2 + \|\eta\|_2^2.$$

It follows that

$$\|T^k \zeta\|_2^2 = \|T^k \xi\|_2^2 + \|T^k \eta\|_2^2 = |\lambda|^{2k} \|\zeta\|_2^2.$$

Since the last term grows without bound as $k \to \infty$, at least one of the two sequences $\left\{\|T^k \xi\|_2\right\}$ and $\left\{\|T^k \eta\|_2\right\}$ must do likewise, contradicting condition (2).

Since (3) clearly implies (2), it suffices to show that (1) implies (3). Suppose therefore that (1) holds. Pick $\epsilon > 0$ such that $|\lambda| + \epsilon < 1$ for any eigenvalue; this can be done since each has modulus strictly less than one. We are free to choose a basis such that each $m \times m$ Jordan block has the form (see Problem 12 of Chapter 8)

$$J_\epsilon = \begin{pmatrix} \lambda & \epsilon & 0 & \cdots & 0 \\ 0 & \lambda & \epsilon & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda \end{pmatrix}. \tag{9.3}$$

On the transformed basis, choose the norm

$$|y| = \sum_{k=0}^{n} |y_k|. \tag{9.4}$$

Here the term $|y_k|$ under the summation sign refers to the modulus of the complex number y_k. This induces a norm $\|\cdot\|$ on the original basis, i.e. if $x = Sy$ then $\|x\| = |y|$. Then for any $x \neq 0$,

$$\|Tx\| = |S^{-1}Tx| = |S^{-1}TSy| = |Jy| \tag{9.5}$$

where J is the full Jordan matrix consisting of a number of blocks like that of equation (9.3). Since $(Jy)_k = \lambda_s y_k$ or $(Jy)_k = \lambda_s y_k + \epsilon y_{k+1}$ for each value of $k = 1, 2, \ldots, n$ and some value of s (where the latter indexes the distinct eigenvalues), it follows that $|Jy| < (\mu_0 + \epsilon)|y| = \mu\|x\|$ where μ_0 is the largest of the moduli of the eigenvalues of T. Since ϵ is so chosen that $\mu = \mu_0 + \epsilon < 1$, $\|Tx\| < \mu\|x\|$ for all x, and this implies (3).

This proves the equivalence of the three statements. □

Under any of the equivalent assumptions (1),(2) and (3) it is clear that the origin is asymptotically stable. On the other hand, if $\lambda| > 1$ for any eigenvalue of T the origin is seen to be unstable by the argument in the first paragraph of the proof above. We may therefore summarize the results of this section in an analog to Theorem (8.3.2) of the preceding chapter:

Theorem 9.1.1 *The origin is asymptotically stable for the linear system (9.2) if the modulus of every eigenvalue is less than one. It is unstable if any eigenvalue has a modulus greater than one.* □

In each of the two cases covered by this theorem, the linear theory is decisive also for the nonlinear case, as we now proceed to show.

9.1.2 Nonlinear stability for maps

We begin with the case when the origin is asymptotically stable for the linearized problem.

Theorem 9.1.2 *Let* 0 *be a fixed point of a discrete dynamical system based on a* C^1 *mapping* $g : \Omega \to \Omega$*. If the eigenvalues of the mapping* $T = g_x(0)$ *are less than one in modulus, then the fixed point is asymptotically stable.*

Proof: Choose a norm so that $\|T\| \leq \mu < 1$. By Taylor's theorem,

$$g(x) = Tx + h(x) \text{ where } \|h(x)\|/\|x\| \to 0 \text{ as } \|x\| \to 0. \tag{9.6}$$

Choose $\epsilon > 0$ so that $\mu + \epsilon < 1$, then choose $\delta > 0$ so that $\|x\| < \delta$ implies that $\|h(x)\| < \epsilon\|x\|$. Then

$$\|x\| < \delta \Rightarrow \|g(x)\| \leq (\|T\| + \epsilon)\|x\| \leq (\mu + \epsilon)\|x\|.$$

This shows that the ball of radius δ is positively invariant, and, since $\mu+\epsilon < 1$, that $\|g^n(x)\| \to 0$ as $n \to \infty$. $\square$

Remarks:

- As usual, if the fixed point is at $p \neq 0$, we make the transformation $x \to x + p$ in order to apply the theorem as stated.
- The nonlinear mapping g is a *contraction mapping* in a neighborhood N of the origin, i.e. $\|g(x) - g(y)\| \leq \nu\|x - y\|$ with $\nu < 1$. This can be verified as follows. By the mean value theorem, $g(x) - g(y) = F(x, y)(x - y)$ where F is continuous for x, y sufficiently small and $F(0, 0) = g_x(0) = T$. The norm of F can be chosen as close to that of T as we desire, so in particular is less than 1 if the neighborhood N is small enough.
- As also in the case of differential equations, the case when the origin of the linearized system is stable but not asymptotically stable is left uncovered by this theorem.

The following instability theorem for nonlinear maps is proved in a manner very similar to that of Theorem 8.4.2 of Chapter 8.

Theorem 9.1.3 *Let* 0 *be a fixed point of a discrete dynamical system based on a* C^1 *mapping* $g : \Omega \to \Omega$. *If any eigenvalue of the mapping* $T = g_x(0)$ *exceeds one in modulus, then the fixed point is unstable.*

Proof: We rewrite the mapping in the form of equation (9.6) above. We next transform to coordinates in which the matrix T takes the Jordan form in which each block has the structure of equation (9.3), and write the transformed matrix in the form

$$S^{-1}TS = B = \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix}$$

so that $B^{(1)}$, of dimension n_1 (say) consists of Jordan blocks whose eigenvalues all exceed one in modulus, whereas none of the eigenvalues of $B^{(2)}$ exceeds one in modulus. Let the dimension of $B^{(2)}$ be n_2 so that $n_1 + n_2 = n$. The mapping, in the new variable $y = S^{-1}x$ takes the form

$$y^{(1)\prime} = B^{(1)}y^{(1)} + k^{(1)}(y^{(1)}, y^{(2)}),$$
$$y^{(2)\prime} = B^{(2)}y^{(2)} + k^{(2)}(y^{(1)}, y^{(2)}).$$

Here the vector function $k(y)$ is obtained from the function $h(x)$ of equation (9.6) by the formula $k(y) = S^{-1}h(Sy)$ and satisfies the same estimate. We may define a norm for $y^{(j)}$ by the formula

$$|y^{(j)}| = \sum_{l=1}^{n_j} |y_l^{(j)}|, \; j = 1, 2.$$

In this formula the vertical bars $|\cdot|$ in the sum on the right-hand side represent the moduli of the (in general) complex numbers $y_l^{(j)}$ whereas those on the left-hand side of the equation indicate the norm of the vector $y^{(j)}$. This is the choice of norm made in equation (8.9) of Chapter 8 and used also in the proof of Theorem 9.1.1. The norm for the full vector $y = (y_1, y_2)^t$ is then $|y| = |y_1| + |y_2|$. This induces a norm for x: $\|x\| = |y|$.

Let $\mu > 1$ be a lower bound for the eigenvalues of B_1. It is easy to see that $|B^{(1)}y^{(1)}| > (\mu - \epsilon)|y^{(1)}|$. Similarly, we find $|B^{(2)}y^{(2)}| \leq (1 + \epsilon)|y^{(2)}|$. By choosing $|y|$ sufficiently small we can ensure that

$$|k(y)| < \epsilon|y| = \epsilon(|y^{(1)}| + |y^{(2)}|).$$

Armed with these estimates, we can now show that the origin is unstable.

Assume, to the contrary, that the origin is stable. Denote by $y(0)$ the initial choice of y, and by $y(1), y(2), \ldots$ subsequent iterations under the mapping $By + k(y)$. According to the assumption of stability, given any $\epsilon > 0$ there exists $\delta > 0$ such that, if $|y(0)| < \delta$, then $|y(n)| < \epsilon$ for all $n = 1, 2, \ldots$. With the aid of the estimates above we find that

$$|y^{(1)}(n+1)| \geq (\mu - \epsilon)|y^{(1)}(n)| - \epsilon(|y^{(1)}(n)| + |y^{(2)}(n)|)$$

and

$$|y^{(2)}(n+1)| \leq (1 + \epsilon)|y^{(2)}(n)| + \epsilon(|y^{(1)}(n)| + |y^{(2)}(n)|),$$

so

$$|y^{(1)}(n+1)| - |y^{(2)}(n+1)| \geq (\mu - 3\epsilon)\,|y^{(1)}(n)| - (1+3\epsilon)\,|y^{(2)}(n)|.$$

We have not as yet specified ϵ: choose it less than $(1/6)(\mu - 1)$. Then

$$|y(n+1)| \geq |y^{(1)}(n+1)| - |y^{(2)}(n+1)| \geq (1+3\epsilon)\left(|y^{(1)}(n)| - |y^{(2)}(n)|\right).$$

Now choosing $|y^{(1)}(0)| - y^{(2)}(0)| > 0$, we find that

$$|y(n)| \geq |y^{(1)}(n)| - |y^{(2)}(n)| \geq (1+3\epsilon)^n\left(|y^{(1)}(0)| - |y^{(2)}(0)|\right),$$

implying that the left-hand side grows without bound as $n \to \infty$. This contradicts the assumption of stability. □

PROBLEM SET 9.1.1

1. Consider the one-dimensional *logistic* map $g(x) = rx(1-x)$ on the interval $I:\ 0 \leq x \leq 1$, where r is a real constant.
   (a) For what values of r does g take I to itself?
   (b) For what values of $r \in R$ does g have only one fixed point? Two fixed points?
   (c) Investigate the stability of the fixed points of part 1b
2. The *standard* map is defined on all of R^2 by the formulas
   $$x' = x + K \sin y, \quad y' = y + x'.$$
   Assume $K > 0$.
   (a) Find the fixed points of this map with y restricted to the interval $[0, 2\pi)$
   (b) What can you say about the stability of these fixed points?
   (c) Show that the standard map preserves area: if G is a region of the xy plane with area A, then its image G' under the map has the same area.
3. The *Henon* map defined on R^2 is
   $$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} x_2 + 1 - ax_1^2 \\ bx_1 \end{pmatrix},$$
   where a and b are real constants.
   (a) Find conditions on a, b so that the Henon map possesses real fixed points.
   (b) Find the expressions for the eigenvalues at the fixed points and determine their stability for the case when $b = 0$.

9.2 Linear systems with periodic coefficients

The focus in dynamical-systems theory is on nonlinear equations but in the present section we regress to linear systems. Our principal result is Floquet's theorem, Theorem 9.2.1 below. We'll need some preliminaries.

Lemma 9.2.1 *Consider the linear, homogeneous system $\dot{x} = A(t)x$ with matrix A continuous on the interval I, and suppose that Φ and Ψ are both fundamental matrix solutions. Then there exists a nonsingular matrix K such that $\Psi(t) = \Phi(t)K$.*

Proof: It is straightforward to check that $(d/dt)\Phi^{-1}\Psi = 0$ on I, so $\Phi^{-1}\Psi$ is constant there. Both matrices are nonsingular so the constant matrix K is likewise nonsingular. □

We shall need to define the logarithm of a nonsingular matrix. This definition is based on the series expansion about the origin of the complex function $\ln(1 + z)$.

Lemma 9.2.2 *Let C be a constant, nonsingular matrix (real or complex). There exists a matrix D, called the logarithm of C, such that $e^D = C$.*

Proof: Let J be the Jordan form of C. If we can find a logarithm matrix D in this case, transformation by the similarity transformation for C gives the required form of the logarithm on the original basis. We seek D with the same block structure as J: $D = \text{diag}(D_0, D_1, \ldots, D_r)$, where D_k is an $n_k \times n_k$ matrix. Then

$$e^D = \begin{pmatrix} e^{D_0} & \cdots & 0 & \\ 0 & e^{D_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{D_r} \end{pmatrix}$$

and the problem is reduced to that of finding D_k such that $e^{D_k} = J_k, k = 0, \ldots, r$. Of these, the simplest is that for $k = 0$, since J_0 is a diagonal matrix with entries $\lambda_0, \ldots, \lambda_p$. Therefore

$$e^{D_0} = \text{diag}(\lambda_0, \ldots, \lambda_p) \Rightarrow D_0 = \text{diag}(\ln \lambda_0, \ldots, \ln \lambda_p).$$

Here the entries are ordinary complex numbers. The logarithms are all defined (though not uniquely) since the eigenvalues of C are nonzero. For $k = 1, \ldots, r$ we need to solve an equation of the form

$$e^{D_k} = \lambda_{p+k} I_{n_k} + Z_{n_k} = \lambda_{p+k}(I_{n_k} + \lambda_{p+k}^{-1} Z_{n_k}), \; k = 0, \ldots, r. \tag{9.7}$$

Now for complex numbers λ and z we have the series expansion

$$\begin{aligned} \ln(\lambda + z) &= \ln[\lambda(1 + \lambda^{-1}z)] = \ln\lambda + \ln(1 + \lambda^{-1}z) \\ &= \ln\lambda + \lambda^{-1}z - \frac{\lambda^{-2}z^2}{2} + \cdots + \frac{(-1)^{m+1}\lambda^{-m}z^m}{m} + \cdots \end{aligned}$$

Thus the obvious candidate for the matrix D_k is the expression:

$$D_k = (\ln\lambda_{p+k})I_{n_k} + \lambda_{p+k}^{-1} Z_{n_k} - \frac{\lambda_{p+k}^{-2} Z_{n_k}^2}{2} + \cdots + \frac{(-1)^{n_k}\lambda_{p+k}^{n_k-1} Z_{n_k}^{n_k-1}}{n_k - 1}.$$

The series terminates because the matrix variable Z is nilpotent, so there is no question regarding its convergence. The identity $e^{\ln(1+z)} = 1 + z$ can be verified for $z \in C$ by expressing $\ln(1+z)$ by its power-series expansion and substituting that into the power-series expansion of the exponential. Since the terms in the expression for D_k commute, this verifies (9.7). □

Remark: As is the case with ordinary logarithms in the complex plane, the logarithm D is not uniquely defined. It is clear that in general, even if C is real, D will be complex. However, it is possible to show that if C is real, then its square C^2 *does* possess a real logarithmic matrix (see [1], Chapter 6, Problem 41).

The object of our consideration is the homogeneous, linear equation with periodic coefficients:

$$\dot{x} = A(t)x, \quad A(t+T) = A(t), \quad -\infty < t < +\infty. \tag{9.8}$$

Without loss of generality, we may suppose $T > 0$. We have the following basic theorem:

Theorem 9.2.1 *(Floquet's Theorem) Let A be continuous on all of R and periodic there with period T. Then if Φ is any fundamental matrix solution of equation* (9.8)*, there exists a matrix P defined on R and periodic there with period T, and a constant matrix Q, such that*

$$\Phi(t) = P(t)e^{Qt}. \tag{9.9}$$

Proof: Let $\Psi(t) = \Phi(t+T)$. Then Ψ is also a fundamental matrix solution to (9.8). It follows from Lemma 9.2.1 that there exists a non-singular, constant matrix M such that $\Phi(T+t) = \Phi(t)M$. The matrix M is called the *Floquet multiplier matrix*. Since M is non-singular, we can find a matrix Q such that $e^{TQ} = M$, T being the period. Now consider the matrix $P(t)$ defined $\forall t$ by the formula $P(t) = \Phi(t)e^{-Qt}$. It remains to be shown that the matrix $P(t)$ is periodic with period T.

$$\begin{aligned} P(t+T) &= \Phi(t+T)e^{-Q(t+T)} = \Phi(t)Me^{-QT}e^{-Qt} \\ &= \Phi(t)e^{-Qt} = P(t). \quad \square \end{aligned}$$

Remarks:

1. We often assume that T is the least period of A, although the theorem does not require this.

2. If the fundamental matrix Φ is chosen so that it reduces to the identity at $t = 0$, then $M = \Phi(T)$.

3. Suppose $A(t)$ is real. The Floquet multiplier matrix M can then also be chosen real, but the matrices P and Q of the theorem will need to be complex in general. Consider, however, $\Phi(t+2T) = \Phi(t+T)M = \Phi(t)M^2$. Since M^2 has a real logarithm (see the remark following Theorem 9.2.2), we infer that (9.9) holds with real P, Q, but now P is of period $2T$. An example follows.

Example 9.2.1

$$A(t) = \begin{pmatrix} \alpha + \beta\cos(t) & \beta\sin(t) - \frac{1}{2} \\ \beta\sin(t) + \frac{1}{2} & \alpha - \beta\cos(t) \end{pmatrix}.$$

One can verify that the fundamental solution is of the correct form with

$$P(t) = \begin{pmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix}.$$

Here P has period 4π whereas A has period 2π. The Floquet multiplier works out to be

$$M = \begin{pmatrix} -e^{(\alpha+\beta)2\pi} & 0 \\ 0 & -e^{(\alpha-\beta)2\pi} \end{pmatrix},$$

which has negative eigenvalues; its square has positive eigenvalues and therefore has a real logarithm. $\square$

It is clear that if all the eigenvalues of the matrix Q have negative real parts then the matrix solution (9.9) of the periodic differential equation (9.8) decays to the origin. It would be easy to prove, using arguments similar to those used for equilibrium points, that if equation (9.8) represented the linearization about a periodic orbit $x = p(t)$ of a system of nonlinear differential equations, then that orbit would be stable. But who said life is easy? In the next section we deal with the subtler case of periodic orbits of autonomous systems.

PROBLEM SET 9.2.1

1. Suppose we knew the matrices $P(t)$ and Q of equation. Introduce the time dependent transformation of variables $x = P(t)y$ to show that y satisfies a linear, homogeneous differential equation with constant coefficients. What are these coefficients?

2. Show that the spectrum of Q is independent of the particular choice of fundamental matrix Φ.

3. For $n = 1$ the equation (9.8) is $du/dt = a(t)u$, where a is $T-$periodic. Find explicit formulas for the function $p(t)$ and the constant q such that the "fundamental" solution has the form of equation (9.9).

4. Let $A(t)$ be the rotation matrix

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

   Find a fundamental matrix $\Phi(t)$ and matrices $P(t)$ and Q such that the representation (9.9) holds.
   Hint: the system can be integrated explicitly in the complex variable $z = x + iy$.

9.3 Orbital Stability

The following example shows that the definition of stability introduced in the preceding chapter is inappropriate for considering the stability of periodic solutions of autonomous systems.

Example 9.3.1 Consider the system

$$\frac{dx}{dt} = -(x^2 + y^2)y, \quad \frac{dy}{dt} = (x^2 + y^2)x; \quad x(0) = 1,\ y(0) = 0.$$

All solutions lie on circles centered at the origin, and the particular solution satisfying the given initial conditions is $(x, y) = (\cos t, \sin t)$, and lies on the circle of radius one. Suppose we consider the solution with the nearby initial conditions $x(0) = 1 + \delta$, $y(0) = 0$, lying on the nearby circle of radius $1 + \delta$. The solution in this case is found to be $(x, y) = (A\cos(A^2 t), A\sin(A^2 t))$, where $A = 1+\delta$. For the solution to be stable according to the definition given in the preceeding chapter, it would be necessary that the square of the Euclidean distance

$$\left(A\cos(A^2 t) - \cos t\right)^2 + \left(A\sin(A^2 t) - \sin t\right)^2$$

remain, for $t > 0$, as small as we please if $|\delta|$ is sufficiently small. However, it is clear that this is not the case for all positive t. For example, for positive integers k and times $t = t_k = (2k + 1/2)\,\pi/\left(A^2 - 1\right)$, the Euclidean distance is $\sqrt{1 + A^2}$. Hence the solution with the given initial data is unstable, although the orbits of the two solutions remain close for all time. □

This discrepancy between the notion of the stability of an equilibrium point as given and applied in Chapter 8, and the stability of a periodic orbit, is not special to the example above but unavoidable, as the following considerations show. Let the autonomous system (9.1) have the periodic solution $x = \psi(t)$ with period T; the linearized system in the neighborhood of the T-periodic solution $x = \psi(t)$ is then given by equation (9.8) with $A(t) = f_x(\psi(t))$. If $\Phi(t)$ is the fundamental matrix solution of this equation reducing to I when $t = 0$, then the Floquet multiplier matrix $M = \Phi(T)$. If all the eigenvalues of M were less than one in modulus, we could easily infer asymptotic stability using a definition of stability like that of definition (8.1.1) of Chapter 8. However, for an autonomous system, it is always the case that M has at least one eigenvalue equal to one. This is easily seen as follows.

Differentiating the equation $d\psi/dt = f[\psi(t)]$ with respect to t, one finds

$$\frac{d}{dt}\left[\frac{d\psi}{dt}\right] = A(t)\frac{d\psi}{dt},$$

i.e. $\xi(t) = d\psi/dt$ is a solution to the linearized equation (9.1). It therefore has a representation $\xi(t) = \Phi(t)c$ for a constant vector c; assuming as we may that $\Phi(0) = I$, we have $c = \xi(0)$. Since this solution is T-periodic, evaluation at $t = T$ shows $Mc = c$, i.e., that the Floquet matrix M has 1 as an eigenvalue, with a corresponding eigenvector

$$c = \xi(0) = \dot{\psi}(0) = f(\psi(0)),$$

which is tangent to the orbit at the point $\psi(0)$. This is responsible for the drift in the direction of the orbit noted in the example, and makes the study of the stability of periodic orbits more delicate than that of equilibrium points.

Let the T-periodic solution $\psi(t)$ determine the orbit γ; i.e. $\gamma = \{x \mid x = \psi(t) \text{ for some } t\}$. Given some choice of norm $\|\cdot\|$, denote distance by d, so that the distance between two points x and y is simply $d(x,y) = \|x-y\|$, whereas the distance $d(x,S)$ from a point x to a set S is the infimum of distances $d(x,y)$ for points $y \in S$, and similarly for the distance $d(S,T)$ between sets S and T. We define *orbital* stability as follows:

Definition 9.3.1 *The periodic solution ψ with orbit γ is* orbitally stable *if, given $\epsilon > 0$ there exists a $\delta > 0$ such that $d[\phi(t,x),\gamma] < \epsilon$ for all $t > 0$ and for all x such that $d[x,\gamma] < \delta$. It is* orbitally asymptotically stable *if it is orbitally stable and (by choosing δ smaller if necessary) $d[\phi(t,x),\gamma] \to 0$ as $t \to \infty$.*

As usual, we'll say that a periodic solution is (orbitally) unstable if it fails to be stable by this definition.

We henceforth take it for granted that it is orbital stability that is principally of interest in realistic problems, and concentrate our attention on it. The strategy for establishing orbital stability is the following. We introduce a hyperplane H of dimension $n-1$ transverse to the direction of the flow. The periodic orbit intercepts this hyperplane in a point p (say) at intervals of time T, where T is the period. Nearby orbits, which are not in general periodic, intercept H in points that are at least initially near p at times separated by intervals near T. If they remain near p, we shall be able to infer that the orbit is stable; if they tend away from p we shall be able to infer instability. The preparatory development and the first two lemmas below do not refer specifically to periodic solutions.

9.4 The Flow Box and the Return Map

Let f be in $C^1(\Omega)$ and suppose $f(p) \neq 0$. Let H be a hyperplane transverse to f at p. That is, $H \subset R^n$ is a linear space of dimension $n-1$ containing the point p and *not* containing the line passing through that point and in the direction of $f(p)$. Any such linear space is described by an equation of the form $h(y) = \text{const}$ for a linear function h. It will be convenient to translate the point p to the origin of coordinates, i.e, put $x = p + y$. Equation (4.1) then becomes

$$\dot{y} = g(y) \quad g(y) \equiv f(p + y). \tag{9.10}$$

We'll continue to denote the domain of g by Ω. The linear function $h(y)$ takes the form $h(y) = c \cdot y$ for some vector $c = (c_1, c_2, ..., c_n)$, and $c \cdot y$ denotes the usual Euclidean inner product: $c \cdot y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$. In terms of the function h, we have:

$$y \in H \Leftrightarrow h(y) = 0 \quad \text{and} \quad h[g(0)] \neq 0. \tag{9.11}$$

Definition: A local section S for g at 0 is a subset of H, open in H, containing 0 and satisfying the condition $h[g(y)] \neq 0$ for all $y \in S$.

Up to the present we have not distinguished between the vector y and its components on a given basis. This is all right if we never change bases, but awkward if we do. In a given, initial choice of coordinates we denote by $\{e_1, e_2, ..., e_n\}$ the standard column n-tuples: e_k has 1 in the kth position and zeros elsewhere. If we choose a new basis $\{\eta_1, \eta_2, ..., \eta_n\}$, we will write

$$y = \sum_{i=1}^{n} y_i e_i = \sum_{i=1}^{n} y_i' \eta_i.$$

If we view y and y' as column vectors, then $y' = Ty$; T, a nonsingular matrix, gives the transformation of coordinates.

For definiteness, we may choose the basis $\{\eta_k\}$ and the hyperplane H as follows: take η_1 in the direction of $g(0)$, and choose the remaining $n-1$ vectors so as to complete η_1 to a basis. Let $H = \text{span}\{\eta_2, \eta_3, ..., \eta_n\}$. In the new coordinate system H is characterized by the condition $y_1' = 0$ and the vector c has the components $(c_1', 0, ..., 0)$, where $c_1' \neq 0$. Define $g'(y') = Tg(T^{-1}y')$; the condition $h[g(y)] \neq 0$ in the definition of S now takes the form $g_1'(y') \neq 0$. Since this holds at $y' = 0$, it holds in some neighborhood of the origin. Therefore the relation

$$S = \{y \in R^n \mid y = y_2'\eta_2 + \ldots + y_n'\eta_n, \sum_{2}^{n} |y_k'| < \epsilon\}.$$

defines the local section S, for some appropriately small choice of ϵ. The differential equation (8.1) becomes

$$\dot{y}' = g'(y') \tag{9.12}$$

and the solution is given by $\phi'(t,p') = T\phi(t,T^{-1}p')$ where $\phi(t,p)$ is now the flow generated by equation (9.10).

We next construct the "flow box" B, an open set in R^n containing a local section S and having the property that $y \in B$ implies that, for sufficiently small σ and a unique t with $|t| < \sigma$, $\phi(t,y) \in S$. Intuitively, the flow box is the geometric object arrived at by allowing S to evolve briefly forward and backward in time under ϕ. We justify this as follows; suppose we solve the equation (9.12) with initial data $y_0 \in S$. The solution $\phi(s,y_0)$ exists on an interval $|s| < \sigma$, for some choice of $\sigma > 0$ and uniformly on S if the latter is small enough, by Theorem 5.4.1 of Chapter 5. This defines a mapping, expressed in the primed coordinates, on the box

$$C = \{(s,\hat{y}') \in R^n \mid |s| < \sigma, \|\hat{y}'\| < \epsilon\},$$

given by $\phi'(s,\hat{y}')$, where the vector $\hat{y}' = (0, y_1', \ldots, y_n')$ and its norm $\|\hat{y}'\|$ may be defined in the usual way as the sum of the absolute values of the components. The image B under this mapping is the set

$$B = \{z' \in R^n \mid z' = \phi'(s,\hat{y}'),\ (s,\hat{y}') \in C\}.$$

This mapping is one-to-one if S and σ are small enough. We see this as follows. The Jacobian matrix of ϕ' with respect to $x' \equiv (s,y')$, written in column structure, is given by the expression

$$\phi'_{x'} = \left(\frac{\partial\phi'}{\partial s}, \frac{\partial\phi'}{\partial y_2'}, \ldots, \frac{\partial\phi'}{\partial y_n'}\right).$$

At the point $y = 0 = y'$, we find

$$\phi_{x'} = \begin{pmatrix} g_1'(0) & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

which is evidently nonsingular in virtue of the assumption that $g_1'(0) \neq 0$. It is therefore a consequence of the implicit-function theorem that, for sufficiently small values of σ and ϵ, the mapping ϕ' from C to B is invertible (a diffeomorphism). The set B is therefore open since it is the inverse image of an open set; it contains the local section S since $\phi'(0,S) = S$. The set C

expressed in the unprimed coordinates is the product of the interval $(-\sigma, \sigma)$ with the section S, and B in these coordinates is the image of this set under the flow. It is now easy to prove the following "flowbox lemma:"

Lemma 9.4.1 *Given equation (9.10) (or equation 9.12) with $g(0) \neq 0$, there is an open set B containing a local section S for g at 0 with the following property: if $y \in B$, there exists a unique $s \in (-\sigma, \sigma)$ such that $\phi(s, y) \in S$. Here σ is a sufficiently small postive number.*

Proof: Suppose $y \in B$. By the previous construction there exists $s \in (-\sigma, \sigma)$ and $y_0 \in S$ such that $\phi(s, y_0) = y$. Solving (9.12) with y as initial data we get

$$\phi(t, \phi(s, y_0)) = \phi(t + s, y_0) = y_0 \in S \quad \text{for} \quad t = -s. \quad \square$$

If the orbit through a point z_0 encounters a transversal H at a point p_0, the orbits through nearby points z will also encounter H at nearby points p, after a time-of-flight that varies smoothly with the initial data. This is the content of the following lemma.

Lemma 9.4.2 *Let $g(0) \neq 0$ and let S be a local section at 0. Suppose the solution ϕ of equation (9.12) satisfies the condition $\phi(t_0, z_0) = 0$ for some $t_0 > 0$ and $z_0 \in \Omega$. Then there is a neighborhood U of z_0 in Ω and a unique map $\tau : U \to R$ which is C^1 on U, such that $\tau(z_0) = t_0$ and $\phi(\tau(z), z) \in S$ for all $z \in U$.*

Proof: The solution $\phi(t, z_0)$ of equation (8.1) exists on the interval $[0, t_0]$ and satisfies $\phi(t_0, z_0) = 0$. It follows from the continuation theorem (Theorem 5.1.5) that the interval can be extended to $[-\nu, t_0 + \nu]$ for some $\nu > 0$, and from the continuity theorem (Theorem 5.4.1) that the solution $\phi(t, z)$ exists and is C^1 on the set $(-\nu, t_0 + \nu] \times U(\delta)$, where $U(\delta)$ is a neighborhood of z_0. Consider the real-valued function $F(t, z) = h[\phi(t, z)]$ on this set. It satisfies the condition $F(t_0, z_0) = 0$, and $F_t(t, z) = h[\phi_t(t, z)] = h[g(\phi(t, z))]$ does not vanish at (t_0, z_0), by virtue of the assumption that $h[g(0)] \neq 0$. From the implicit-function theorem we infer that there exists a neighborhood U of z_0 such that, for any $z \in U$ there exists a unique $t = \tau(z)$ reducing to t_0 at $z = z_0$, which is C^1 for $z \in U$. If U is chosen small enough, $\phi(\tau(z), z) \in S$.

Corollary 9.4.1 *The gradient of the function τ satisfies*

$$\nabla \tau|_{z=z_0} = \frac{-\nabla\left(h\left(\phi\right)\right)|_{z=z_0}}{h[g(0)]}. \tag{9.13}$$

Proof: The derivatives of the C^1 function τ may be calculated by implicit differentiation. Differentiate the expression $h[\phi(\tau(z), z)] = c \cdot \phi[\tau(z), z]$ with respect to z. In index notation using the summation convention one obtains, recalling that $h(y) = c_k y_k$,

$$\frac{\partial h}{\partial y_k}\left(\frac{\partial \phi_k}{\partial z_l} + g_k\left(0\right)\frac{\partial \tau}{\partial z_l}\right) = c_k\left(\frac{\partial \phi_k}{\partial z_l} + g_k\left(0\right)\frac{\partial \tau}{\partial z_l}\right) = 0,$$

or

$$\frac{\partial \tau}{\partial z_l} = -\frac{c_k \phi_{k,l}}{c_k g_k\left(0\right)} \tag{9.14}$$

at $z = z_0$; this agrees with the formula (9.13) above. □

We note in passing that in the primed coordinates adopted above, the formula for the derivative for $\tau'\left(z'\right) = \tau\left(z\right)$ simplifies to

$$\frac{\partial \tau'}{\partial z'_k} = -\frac{\phi'_{1,k}}{g'_1\left(0\right)}. \tag{9.15}$$

Let γ be a periodic orbit of of the system (9.10) with least period T. Construct a local section S transverse to g at $y = 0$. Lemma 9.4.2 applies with $t_0 = T$ and $z_0 = 0$: there is a neighborhood U of the origin and a uniquely determined C^1 function $t = \tau(z)$ on U reducing to T when $z = 0$, such that $\phi[\tau(z), z] \in S$ for $z \in U$. Then if $S_0 = S \cap U$, the mapping $G : S_0 \to S$ given by

$$G(y) \equiv \phi[\tau(y), y] \tag{9.16}$$

defines the *return map* or *Poincaré map*.

Lemma 9.4.3 *Suppose G maps S_0 to itself, and for some $y_0 \in S_0$, $G^m(y_0) \to 0$ as $m \to \infty$. Then $d[\phi(t, y_0), \gamma] \to 0$ as $t \to \infty$.*

Proof: Let $y_m = G^m(y_0)$. Then $y_m \in S_0$ for all m. Let $T_m = \tau(y_m)$ and note that $y_m \to 0$ implies $T_m \to T$, the least period of γ, according to Lemma 9.4.2. Let $T*$ be an upper bound for $\{T_m\}$ and consider the difference $\|\phi(t, y) - \phi(t, 0)\|$ on $[0, T*] \times K$, where K is a compact subset of S_0 containing y_m for all sufficiently large m. This difference can be made arbitrarily small if $\|y\|$ is chosen sufficiently small, uniformly with respect to $t \in [0, T*]$, by the uniform continuity of ϕ on this compact set. It follows that $\|\phi(t, y_m) - \phi(t, 0)\| \to 0$ as $m \to \infty$ uniformly for $t \in [0, T^*]$. Now consider $d\left(\phi\left(t, y_0\right), \gamma\right)$. When

t lies in the interval $\sum_1^n T_k \leq t < \sum_1^{n+1} T_k$ we may $t = \sum_1^n T_k + s$. Then $s \in [0, T_{n+1}] \subset [0, T^*]$. On this interval,

$$d[\phi(t,y),\gamma] \equiv \inf_{z\in\gamma} \|\phi(t,y) - z\| \leq \|\phi(s,y_n) - \phi(s,0)\|$$

since $\phi(t,y) = \phi(s,y_n)$ and $\phi(s,0) \in \gamma$. Since we have shown that this last expression can be made arbitrarily small if n is chosen sufficiently large, the conclusion follows. □

9.5 The Stability Theorems

We have previously observed that the vector $f(p) = g(0)$ is an eigenvector of the Floquet multiplier matrix M with eigenvalue one. Choosing the unit vector η_1 in the direction of $g(0)$ and completing η_1 to a basis as above, we easily find that the Floquet multiplier matrix, expressed in these (primed) coordinates, takes the form

$$M' = \begin{pmatrix} 1 & m^t \\ 0 & B \end{pmatrix} \tag{9.17}$$

where $m^t = (M'_{12}, \ldots, M'_{1n})$ is a row vector and B represents the lower right-hand $(n-1)\times(n-1)$ block of the matrix M'. The remaining eigenvalues of M are those of the smaller matrix B and are evidently those that determine linear instability or stability of the orbit. In fact, we can make the transformation of variables such that the row-vector m^t vanishes; transformation to real-canonical form will achieve this. In the following lemma we relate this block to the Poincaré map. Recall that the fundamental matrix solution of the linearization of equation (9.10) is the Jacobian matrix of its flow map and therefore that the Floquet multiplier matrix of the T-periodic solution passing through the origin is given by $M = \phi_y(T,0)$.

Lemma 9.5.1 *Let $M' = \phi'_{y'}(T,0) = \Phi'(T)$ be the Floquet matrix evaluated at the point $y' = 0$ of the T-periodic orbit γ. Then if $G' : S_0 \to S$ is the Poincaré map defined on the local section S_0, its Jacobian matrix $G'_{\hat{y}'}$ is given by the lower right-hand $(n-1)\times(n-1)$ block B of M'.*

Proof: We use the primed coordinate system in which η_1 is in the direction of $g(0)$ and therefore $g'_i(0) = 0$ if $i \neq 1$. By its definition, in the primed coordinates, $G'_1 = 0$. For $i, j = 2, \ldots, n$

$$\frac{\partial G_i'}{\partial y_j'} = \frac{\partial \phi_i'[\tau(y'), y']}{\partial t} \cdot \frac{\partial \tau'}{\partial y_j'} + \frac{\partial \phi_i'[\tau(y'), y']}{\partial y_j'}$$

implying that at $y' = 0$

$$\frac{\partial G_i'}{\partial y_j'} = g_i'(0) \cdot \left(\frac{\partial \tau'}{\partial y_j'} \right)_{y=0} + M_{ij}',$$

from which the conclusion follows. □

Corollary 9.5.1 *Suppose that $n-1$ eigenvalues of the Floquet multiplier matrix M are less than 1 in modulus. Then the point 0 is an asymptotically stable fixed point of the Poincaré map G.*

Proof: All eigenvalues of the linearization of G at the origin are less than 1 in modulus, so the conclusion follows from Theorem 9.1.2. □

The lemma immediately implies orbital instability when at least one eigenvalue of M exceeds one in modulus.

Theorem 9.5.1 *Suppose that at least one eigenvalue of M exceeds one in modulus. Then the closed orbit γ is unstable.*

Proof: By Theorem 9.1.3, there are iterates by G of points p, initially arbitrarily close to γ, that lie outside some fixed neighborhood of 0. Since all these points lie on the orbit $\phi(t, p)$, the latter recedes from γ by at least this amount, and consequently γ cannot be stable. □

The spectrum of the Floquet matrix M, or equivalently the matrix Q in the representation $\Phi = Pe^{Qt}$, is important in deciding stability or instability of the orbit. It must then be the case that these spectra are determined by the orbit, and not by the particular point of the orbit where we decide to erect a local section. Suppose we create a local section at p; then $M(p) = \phi_x(T, p)$ and for any other point q of the orbit, $M(q) = \phi_x(T, q)$. Now $q = \phi(s, p)$ for some $s \in [0, T)$. It is now a simple matter to show that the matrices $M(p)$ and $M(q)$ are related by a similarity transformation, and therefore have the same spectra.

The basic stability result for periodic solutions is the following.

Theorem 9.5.2 *Let ψ be a T-periodic solution of the autonomous system (9.10) with orbit γ. Let p be a point of γ and suppose $n-1$ eigenvalues of the Floquet multiplier matrix $\phi_x(T,p)$ are less than one in modulus. Then γ is orbitally, asymptotically stable.*

Proof: Choose a local section S of the vector field at p in the following way. First, choose S small enough so that the Poincare map G is contracting[1] on S. Next, suppose $\epsilon > 0$ is prescribed as in the definition of stability. Choose S small enough so that if $x \in S$, then $\mathrm{d}[\phi(t,x),\gamma] < \epsilon$ for $t \in [-T^*, +T^*]$ where T^* is chosen, as in Lemma 9.4.3, as an upper bound for the return times. This ensures that an orbit starting in S remains inside a tube of radius ϵ until it returns to S, and therefore for all subsequent times.

We now need only make sure that an orbit starting near an arbitrary point of γ intercepts this small section S. That is, we need to choose $\delta > 0$ such that $\mathrm{d}[x,\gamma] < \delta$ implies that $\mathrm{d}[\phi(t,x),\gamma] < \epsilon$ until some time t^* such that $\phi(t^*,x) \in S$. A solution through a point q suficiently close to a point $q*$ of γ will, by continuity, remain arbitrarily close to γ on a fixed, finite time interval; we may choose for the latter $2T$. This solution will intercept S_0 at an earlier time by Lemma 9.4.2 □

This theorem shows that nearby orbits approach the periodic orbit γ, but in fact each nearby orbital phase point approaches a particular phase point of γ.

Theorem 9.5.3 *Asymptotic Completeness Under the assumptions of Theorem 9.5.1, if x is sufficiently close to γ, then $\phi(t,x) \to \phi(t,p)$ for some $p \in \gamma$.*

Proof: Using the nototion of Lemma 9.4.3, consider

$$T_n - T \,=\, \tau(y_n) - T \,=\, \tau(0) + \nabla\tau(0)\cdot y_n + o(|y_n|) - T \,=\, O(|y_n|)$$

where we have observed that $\nabla\tau(0)$ is continuous in a neighborhood of the fixed point. Further, $\|y_n\| \le \nu^n\|y\|$ for $\nu \in [0,1)$ (G is contracting) and so we find $\|T_n - T\| \le K\nu^n\|y\|$ for large enough n and some suitable constant K. Therefore, by comparison with the geometric series, the series $\sum_n (T_n - T)$ converges; denote this sum by θ'. Let $\theta = \theta' (\mathrm{mod}\ T)$, so that it lies in the

[1] We measure distance by means of the norm for which this is true, as in Lemma 9.1.1.

interval $[0, T)$. Let $p = \phi(-\theta, 0)$. Then, once again using notation from Lemma 9.4.3

$$\begin{aligned}
\|\phi(t,y) - \phi(t,p)\| &= \|\phi(s,y_n) - \phi(s + \sum_{k=1}^{n}(T_k - T), p)\| \\
&\quad \text{where we use}\, \phi(\eta - nT, p) = \phi(\eta, p) \\
&= \|\phi(s,y_n) - \phi(s + \theta + \gamma_n, p)\| \\
&\quad \text{where}\, \gamma_n = \sum_{k=1}^{n}(T_k - T) - \theta; \quad \gamma_n \to 0\, \text{as}\, n \to \infty \\
&\leq \|\phi(s,y_n) - \phi(s+\theta, p)\| \\
&\qquad + \|\phi(s+\theta, p) - \phi(s+\theta+\gamma_n, p)\|
\end{aligned}$$

Note that $\phi(s+\theta, p) = \phi(s+\theta, \phi(-\theta, 0)) = \phi(s, 0)$. Taking limits as $n \to \infty$ we find $\phi(s, y_n) \to \phi(s, 0)$ and $(s+\theta+\gamma_n) \to (s+\theta)$. Thus the desired claim holds. □.

We have the following corollary.

Corollary 9.5.2 $\|\phi(t+T, y) - \phi(t, y)\| \to 0$ *as* $t \to \infty$.

Proof: By the preceeding result, we know $\|\phi(t, y) - \phi(t, p)\| < \epsilon$ for $t > T^*$ and some $p \in \gamma$. Then

$$\begin{aligned}
\|\phi(t+T,y) - \phi(t,y)\| &\leq \|\phi(t+T,y) - \phi(t+T,p)\| \\
&\qquad + \|\phi(t+T,p) - \phi(t,y)\| \\
&\leq \|\phi(t+T,y) - \phi(t+T,p)\| \\
&\qquad + \|\phi(t,p) - \phi(t,y)\| \\
&\leq 2\epsilon \quad \text{when}\, t > T^*. \qquad \square
\end{aligned}$$

Remarks:

(1) The corollary is a consequence of asymptotic stability alone, and does not depend on asymptotic completeness. If $\epsilon > 0$ is given and δ is chosen small enough, then $\|y - z\| < \delta$ implies $\|\phi(t, z) - \phi(t, y)\| < \epsilon$ for all $|t| < T$. Since $\phi(t, y) \to \gamma$, we may find p (depending on t) in γ such that $\|\phi(t, y) - p\| < \delta$. Then

$$\begin{aligned}
\|\phi(t+T,y) - \phi(t,y)\| &\leq \|\phi(t+T,y) - p\| + \|p - \phi(t,y)\| \\
&= \|\phi(T,\phi(t,y)) - \phi(T,p)\| + \|p - \phi(t,y)\| \\
&< \epsilon + \delta
\end{aligned}$$

(2) The estimate used in the theorem can be improved:

$$T_n - T = \tau(y_n) - T = \tau(0) + \nabla\tau(0) \cdot y_n + o(|y_n|) - T = o(|y_n|).$$

We claim that $\nabla\tau(0) \cdot y_n = 0$. Recall that in the primed coordinates we have

$$\nabla\tau(0) \cdot y_n \propto \frac{y_1'}{g_1'(0)}$$

which vanishes on H. It follows that in a suitable neighborhood of the fixed point, $|T_n - T| \leq \epsilon\nu^n|y|$ where ϵ is as small as we please.

Example 9.5.1

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2), \quad \dot{z} = az.$$

A periodic solution is given by $\psi(t) = (\cos t, \sin t, 0)^t$. Linearizing about this solutions yields the periodic matrix

$$A(t) = f_x(\psi(t)) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t & 0 \\ 1 - \sin 2t & -2\sin^2 t & 0 \\ 0 & 0 & a \end{pmatrix}.$$

A fundamental solution for the linearized equation $\dot{x} = A(t)x$ is $\Phi(t) = P(t)e^{Qt}$, where

$$P(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

It is immediately seen that the orbit is orbitally, asymptotically stable if $a < 0$ and orbitally unstable if $a > 0$. □

Bibliography

[1] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations*. New York: McGraw-Hill, 1955.

[2] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. Springer-Verlaf, New York, 1983.

[3] Einar Hille. *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company, 1969.

[4] M. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York, 1974.

[5] S. MacLane and G. Birkhoff. *Algebra*. New York: Macmillan, 1967.