On the Weakly Nonlinear Development of the Elliptic Instability

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Abstract

Results are presented for the outcome of the elliptic instability, investigated by methods of dynamical-systems theory. Finite-dimensional nonlinear systems are obtained through Galerkin truncation using a systematic truncation criterion that exactly captures any fluid behavior that can also be captured via amplitude expansions. Six different regions of parameter space are explored, corresponding to linear instabilities of different symmetries and temporal types (“steady” or “oscillatory”). Four different kinds of bifurcation behavior are found among the six cases considered. One of these equilibrates at small amplitude, and the others do not, but depart significantly from the unstable equilibrium solution.
1 Introduction

The term elliptic instability refers to the simple laminar flow

\[ \mathbf{V} = K \left( \frac{a_1}{a_2} x_2, -\frac{a_2}{a_1} x_1, 0 \right) \]

which is a steady solution of the Euler equations of incompressible, inviscid fluid dynamics if the boundary has an elliptical cross-section with semi-axes \( a_1 \) and \( a_2 \), or of the equations for viscous flow in the absence of boundaries; \( K \) is an arbitrary constant. The streamlines are then also ellipses in planes \( x_3 = \text{constant} \), with semi-axes proportional to \( a_1 \) and \( a_2 \). This velocity field becomes that of rigid-body rotation if \( a_2 = a_1 \) and is then stable to small disturbances in most of the applications in which it arises. In the context of an unbounded domain, it is unstable to three-dimensional disturbances whenever \( a_2 \neq a_1 \), without any other restriction; it is sometimes said to be “universally” unstable because of this. In principle, this universal, or unconditional, instability should occur also in bounded domains for disturbances of sufficiently small wavelength; this has been verified in a particular case. In practice the smallest wavelengths are damped by friction and therefore normal modes of large to moderate spatial scales are of greater interest. The normal modes may also be unstable, but are not universally so: for any one of them, there is a wedge-shaped region of instability in the parameter space; see also Figure 1 below).

Since many mathematical analyses of flows in approximately circular domains are carried out under the simplifying assumption of exact circular symmetry, these analyses may suppress the elliptic instability and lead to false conclusions. A number of physical problems have therefore been reassessed in light of the discovery of this instability. In these reassessments, information regarding the nonlinear development is needed. That is the objective of the work presented in this paper.

There are numerous (in principle infinitely many) regions in parameter space where the flow (1) is unstable. We have so far investigated six of them. The approach is to construct finite-dimensional dynamical systems capturing certain features of the infinite-dimensional equations of fluid dynamics in a weakly non-linear approximation. One earlier dynamical-systems approach to the nonlinear problem is based on normal-forms theory for Hamiltonian dynamical systems; other approaches are mentioned in §5 below. Our finite-dimensional systems are not Hamiltonian, so a direct comparison with the Hamiltonian normal-forms theory is not possible, but we are able to make comparisons nonetheless and we find qualitative agreement in one of the six cases we have thus far considered.

In this paper we state theorems without proof and quote without details a number of results of lengthy calculations and of numerical computations; these lengthy mathematical and computational issues will be presented elsewhere.

Most of the laboratory experiments have been conducted using elliptic cylinders filled with water, but some have used ellipsoids; qualitatively, and even semi-quantitatively, the results are similar. In the theory described below, it is convenient to take the container to be an ellipsoid.
2 Description of the Problem

The underlying problem is the following. Consider the Euler equations governing an inviscid, incompressible fluid filling a domain $D \in \mathbb{R}^3$:

$$\begin{align*}
u_t + u \cdot \nabla u &= -\nabla p, \quad \text{div } u = 0 \quad \text{in } D, \\
u \cdot n &= 0 \quad \text{on } \partial D.
\end{align*}$$

The domain

$$D = \left\{ x : \sum_{i=1}^{3} x_i^2/a_i^2 < 1 \right\}$$

is fixed in the inertial (laboratory) frame. In a laboratory experiment, the fluid is subject to the force of gravity, but the effect of that may be subsumed in the gradient (pressure) term of equation (2). The linear stability theory for this flow reveals regions of parameter space where one or another normal mode is unstable. The instability is of parametric type familiar from Hamiltonian dynamics: its onset is marked by parameter values at which the oscillation frequencies of a pair of normal modes coincide.

If the aspect ratios are combined in the forms

$$\begin{align*}
\epsilon &= \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}, \\
\mu &= \sqrt{\frac{2a_3^2}{a_1^2 + a_2^2}}
\end{align*}$$

so that $\epsilon$ represents departure from axial symmetry and $\mu$ represents the height, then some of the regions of instability are as shown in Figure 1.

Models for the behavior of this fluid system have been constructed by Guckenheimer and Mahalov$^7$ and by Knobloch et. al$^8$. The Hamiltonian

$$H = (-\mu + \epsilon)q^2/2 - (\mu + \epsilon)p^2/2 + (q^2 + p^2)^2/2$$

of one degree of freedom is derived in the first of these papers (in a different notation) on grounds of symmetry. The system for which this expression is the Hamiltonian has, for all values of the parameters, the origin as an equilibrium point. An easy calculation shows that the instability region of the origin, depicted in the $\epsilon - \mu$ parameter plane, is a wedge like those found in the actual fluid system and depicted in Figure 1. Moreover, one can solve Hamilton’s equations for equilibrium solutions explicitly in this model, thereby constructing the bifurcation of further equilibrium solutions at those critical points in this parameter plane where an eigenvalue $\lambda$ of the linearized stability problem vanishes (see Figure 2a below). Since the equations of fluid dynamics play no explicit role in the derivation of the Hamiltonian (6), one may inquire (a) does the behavior determined from this system in fact reflect that of the fluid and, if so, (b) how should one interpret the dynamical variables $q, p$? We address these and related questions below.
3 Finite-dimensional systems

Let \( u = V + v \) where \( V \) is the unperturbed flow (1). The Euler equations become

\[
\begin{align*}
\frac{\partial v}{\partial t} + L v + v \cdot \nabla v &= -\nabla p, \\
\text{div } v &= 0 \text{ in } D \text{ and } v \cdot n = 0 \text{ on } \partial D.
\end{align*}
\]

Here the operator \( L \) is given by the formula

\[
L v = (V \cdot \nabla) v + (v \cdot \nabla) V
\]

and \( p \) represents the perturbation of the pressure. One can construct finite-dimensional dynamical systems from these equations with the aid of a basis \( \{ \eta_i \} \) of vector fields satisfying (8): for an \( n \)-term truncation \( v(x,t) = \sum_{i=1}^n c_i(t) \eta_i(x) \), one finds

\[
M \dot{c} + A c + f(c) = 0, \quad c \in \mathbb{R}^n,
\]

where

\[
M_{ij} = (\eta_i, \eta_j), \quad A_{ij} = (\eta_i, L \eta_j), \quad f_i = (\eta_i, \sum_{k} c_{ik} \eta_k \cdot \nabla \eta_k),
\]

and the inner product is defined by the formula

\[
(\eta, \xi) = \int_D \eta(x) \cdot \xi(x) \, dx.
\]

The usefulness of the dynamical system (10) depends on the nature of the basis and on a criterion for truncation. Choosing either badly will result in a system from which no reliable information regarding the fluid system may be extracted. An appropriate basis of vector polynomials satisfying the conditions (8) indeed exists\(^\text{10} \). The issue of the truncation criterion is addressed in Theorem 3.3 below.

The linear theory is exactly reducible to a finite-dimensional problem in virtue of the following theorem:

**Theorem 3.1** If \( \eta \) is a vector polynomial of degree \( n \), then so also is the projection of \( L \eta \) on the linear space defined by the conditions (8).

This theorem depends on the linear nature of the unperturbed velocity field (1)\(^\text{11} \) and on choice of an ellipsoid as the fluid domain.

The space of three-component vector fields can be decomposed into subspaces according to symmetry under the invariance group of the ellipsoid, as in Table 1.

Any vector field can be expressed as a unique linear combination of vectors in these subspaces (much as an arbitrary scalar function can be expressed as the sum of an even and an odd function). Each of the basis vectors \( \eta_i \) lies in one or another of these subspaces. The study of the spectrum of the linear operator is then further simplified by the following:
Theorem 3.2 Any eigenfunction of the linear problem may be assumed to lie in one of the classes

\[ 1 \oplus 3c, \quad 2a \oplus 2b, \quad 3a \oplus 3b, \quad 2c \oplus 4. \]

Theorems 3.1 and 3.2 may be combined to conclude that the linear operator \( L \) acts invariantly on finite-dimensional spaces spanned by basis vectors of polynomials of degree less than or equal to some fixed value and lying in one of the four subspaces of Theorem 3.2.

The parameter domains of instability for the basic flow (1) that are indicated in Figure 1 were obtained by considering the linear stability equations obtained from equation (7) above by ignoring the nonlinear term, and solving this linear equation in relatively low-order polynomials (of degree not greater than three), in each of the symmetry classes of Theorem 3.2.

The exact solvability of the linear problem does not extend, of course, to the full nonlinear problem. It does, however, extend to certain weakly nonlinear approximations to the nonlinear problem. Specifically, suppose we seek an approximation to the solution of equation (7) as a formal expansion

\[ v = a v^{(1)} + a^2 v^{(2)} + \ldots, \tag{13} \]

where \( a \) is a small parameter measuring the amplitude of the perturbation. The equation for \( v^{(1)} \) will be the linear equation. Suppose it has a solution of degree \( N^{(1)} \). Then it is easy to see that the solution for \( v^{(2)} \) will be a polynomial of degree \( N^{(2)} = 2N^{(1)} - 1 \) and therefore expressible as a finite linear combination of the basis vectors \( \eta \). Successive approximations are likewise expressible as finite linear combinations, the number of basis vectors needed increasing (in a predictable manner) with the number of terms retained in the amplitude expansion (13). This leads to the next result, which provides our truncation criterion.

Theorem 3.3 For an approximate solution of the system obtained from the amplitude expansion (13) up to a given level \( a^k \), there is a finite set of basis vectors such that the Galerkin truncation (10) based on this set has a solution exactly representing that approximate solution.

The nonlinear calculations are further simplified by exploiting the following nonlinear generalization of Theorem 3.2:

Theorem 3.4 The following classes are invariant for the full nonlinear system (7):

\[ 1 \oplus 3c, \quad 1 \oplus 3c \oplus 2a \oplus 2b, \quad 1 \oplus 3c \oplus 3a \oplus 3b, \quad 1 \oplus 3c \oplus 2c \oplus 4. \]

The precise set of basis vectors occurring in an amplitude expansion appropriate to a given linear instability is therefore known \emph{a priori} on the basis of the linear theory, and it is that set that is used in the Galerkin truncation. The dynamical system represented by our Galerkin truncation therefore captures any
bifurcation from the solution (1) that can be obtained from Poincaré-Lindstedt expansions, or other amplitude expansions, of degree not greater than $k$ in the dependent variable; of course, the dynamical system contains much richer dynamics than that associated with the bifurcation theory. The sizes of the basis sets, and therefore the dimensions of the dynamical systems, increase with $k$. We have so far limited their sizes by restricting the amplitude expansion to $k = 2$ and, in some cases, to $k = 3$, which correspond to the kinds of normal forms that have been considered elsewhere.

Of considerable usefulness in the analysis of the equations (10) has been the property of reversibility:

**Theorem 3.5** For the system (10) there exists a matrix $R$ which anticommutes with $L$ and with $f$ and satisfies the condition $R^2 = I$.

Reversible systems share certain properties with Hamiltonian systems, and Theorem 3.5 has allowed us to exploit some of these properties.

4 Results

We have carried out numerical calculations and, in some cases bifurcation analyses, in parameter regimes corresponding to linear instabilities belonging to each of the four symmetry classes indicated in Theorem 3.2, including three belonging to class $3a \oplus 3b$. It’s convenient to distinguish between “steady” onset, by which we will mean that for which the onset of instability is indicated by the vanishing of an eigenvalue, and “oscillatory” onset, for which the onset of instability is indicated by an eigenvalue of nonvanishing imaginary part. The six cases we have considered include three of each.

4.1 Steady onsets

In each of the three cases of steady onset, we have carried out numerical calculations and detailed bifurcation analyses. We describe them separately.

4.1.1 $2a \oplus 2b$

The instability wedge marked $F$ in Figure 1 belongs to the class $2a \oplus 2b$. The behavior of the weakly nonlinear system is described qualitatively by the Hamiltonian (6) in the following sense. If one plots equilibrium solutions of the system with that Hamiltonian in $qp\mu$ space for fixed $\epsilon > 0$, one finds the bifurcation diagram of Figure 2a: the zero solution exists for all $\mu$, becomes unstable at $\mu = -\epsilon$ where nonzero solutions branch supercritically (stably) in the $q\mu$ plane, then becomes stable again where nonzero solutions branch supercritically (unstably) in the $p\mu$ plane. The bifurcation results for our Galerkin-truncated system, further confirmed by numerical computations, agrees qualitatively with the branching behavior described above: we find a supercritical pitchfork at the left end of wedge $F$ and a second supercritical pitchfork at the right end.
The critical eigenvectors at these ends lie in different symmetry classes (2b and 2a respectively). We can roughly identify the variables q and p of the model Hamiltonian with the coefficients of the linear eigenvectors.

Laboratory experiments have found solutions ('two-vortex' motions\(^6\)) that are well described by the nonlinear solutions obtained in the present case.

4.1.2 3a ⊕ 3b, case 1

The point marked A in Figure 1 refers to a wedge of linear instability for which the eigenfunctions are in the classes 3a ⊕ 3b. For \( \mu \) increasing and for fixed \( \epsilon > 0 \), bifurcation theory predicts a subcritical pitchfork bifurcation at the left edge of the wedge, and this is confirmed by numerical calculations. This is so far consistent with the normal-forms Hamiltonian (6), if the sign of \( \mu \) is reversed there. But then one should also expect a subcritical pitchfork at the right of the wedge, whereas we find a supercritical pitchfork in the dynamical system obtained from the Galerkin truncation. This behavior is therefore qualitatively distinct from that predicted by the model Hamiltonian.

4.1.3 3a ⊕ 3b, case 2

The point marked D likewise refers to a linear eigenfunction in class 3a ⊕ 3b, but the behavior in the nonlinear regime, which can be determined exactly, is now quite different. The reason that the behavior can be determined exactly is the following. There is a classical, exact solution of the system (2,3) of the form

\[
v = c_1(t)\eta_1(x) + c_2(t)\eta_2(x) + c_3(t)\eta_3(x)
\]

(14)

where

\[
\eta_1(x) = (0, a_2x_3/a_3, -a_3x_2/a_2), \ldots
\]

and the dots indicate two further vector fields obtained by cyclic permutations; these three fields are among the basis vectors used, and the vector field \( \eta_3 \) describes the basic, unperturbed velocity field (1). The velocity field (14) is an exact solution of the fluid-dynamical equations (2) if and only if the three coefficients \( c_i(t) \) satisfy the equations of the dynamics of a rigid body

\[
I_1\dot{c}_1 = (I_2 - I_3)c_2c_3, \ldots
\]

wherein the coefficients play the role of the angular-velocity components, and the moments of inertia are

\[
I_1 = a_1^2 + a_2^2, \ldots
\]

Since the steady flow (1) is obtained by setting \( c_1 = c_2 = 0 \) and \( c_3 = K \), its stability to perturbations restricted to the three-dimensional subspace spanned by \( \eta_1, \eta_2, \eta_3 \) can be described completely in terms of the known dynamics of the Euler rigid-body equations; the unstable wedge marked D in fact represents the domain where the \( x_3 \)-axis is the intermediate axis, making \( I_3 \) the intermediate moment of inertia. But the nature of the nonlinear motion near the onset of instability is now quite different from that obtained from the model\(^7\). It is a global bifurcation in the sense that there are no solutions close to the solution (1) once the parameter lies in the unstable regime \( (\mu > \mu_{\text{crit}}) \), i.e., the amplitude of the perturbation is independent of \( \mu - \mu_c \) (by contrast, the amplitude of the

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perturbation described in paragraph 4.1.1 above is limited by \( \sqrt{\mu - \mu_{\text{crit}}} \). The linearized equations have a triple zero eigenvalue at \( \mu_c \). The Euler rigid-body equations possess certain integrals of the motion and, when the system is reduced by the use of one of these, there is a double zero eigenvalue and the corresponding normal form is of Takens-Bogdanov type, but not of the generic form: it has a symmetry which indeed allows for solutions leaving a neighborhood of the origin\(^\text{13}\). The apparent high degree of degeneracy of this problem is misleading; it is in fact of codimension one when one takes into consideration the reversible and conservative nature of the dynamics.

4.2 Oscillatory onsets

The wedges marked \( B, C \) and \( E \), belonging to symmetry classes \( 3a \oplus 3b, 2c \oplus 4 \) and \( 1 \oplus 3c \), respectively, are all characterized by the onset of instability through oscillations of increasing amplitude. We’ll group these three cases together, since they have similar characteristics. In all cases the nonlinear behavior has been determined by numerical integration of the corresponding Galerkin system, since specific bifurcation analyses of these systems have not been carried out. In all three cases it appears that, in the parameter regimes of linear instability, the bifurcations are global: orbits leave a neighborhood of the equilibrium solution for arbitrarily small, positive values of \( \mu - \mu_{\text{crit}} \).

While the kind of specific normal-form analysis needed to confirm these numerical results has not yet been carried out, generic normal-forms analyses have been carried out for some reversible dynamical systems having critical eigenvalues like those of our oscillatory onsets. In particular, the system of dimension four is described in Iooss et. al\(^\text{14}\). The normal form found there is integrable, with two phases (\( \psi_0 \) and \( \psi_1 \)) and two amplitudes (\( r_0 \) and \( r_1 \)). The behaviors found in this normal-forms analysis are depicted, in projection on the \( r_0r_1 \)-plane, at the bottom in Figure 2. Closed loops in this plane connote bounded (not necessarily periodic!) solutions and are seen to occur when the parameter \( \kappa < 0 \) (this “parameter” depends also on the initial data). There is qualitative agreement between the numerical integrations and the normal-forms predictions. In particular, whereas in the postcritical regime all of our numerically calculated orbits leave a neighborhood of the origin (\( \kappa > 0 \)), in the subcritical regime those with sufficiently small initial data remain near the origin, whereas those with larger (but not very large!) initial data again leave a neighborhood of the origin (\( \kappa < 0 \)).

5 Discussion

The dynamical systems of the form (10) are derived from the fluid-dynamical equations, but they are only of interest if they in fact reflect the behavior of the fluid. We have some information on how well we are achieving this, based on certain diagnostics: constants of the motion of the fluid system should also be preserved, at least approximately, by the Galerkin systems (10). The kinetic
energy of the fluid system is a constant of the motion, and it translates into a function of the phase space variables $c$ of equation (10). However, it is known\textsuperscript{15} that this function is exactly conserved by the latter system without regard to the validity of the approximation. Therefore the energy is not a useful diagnostic of the validity of the Galerkin truncation.

Kelvin’s circulation theorem provides (an overabundance of) constants of the motion. We have thus far exploited this in one case: in the symmetry class $1 \oplus 3c$, for which the equator of the ellipsoid is an invariant contour, the circulation integral about the equator becomes a constant of the motion of the Euler equations on the same footing as the energy equation (i.e., not requiring any Lagrangian information for its evaluation). We have used this as a diagnostic. In the unstable regime for this class of functions, an initially small-amplitude perturbation grows and finally saturates. The equatorial circulation based on a fifty-six function expansion continues to be conserved until about the time saturation is reached, well beyond the heuristic limit provided by the truncation criterion, thereby providing reason to believe that the dynamical systems provide qualitative accuracy beyond this limit. After saturation has been reached, the constancy of the circulation subsequently deteriorates. In Galerkin truncations involving eight, twenty-five and fifty-six functions (corresponding to exactly capturing amplitude expansions of degrees one, two and three), the conservation of the equatorial circulation persists successively longer, as one would expect. This is shown in Figure 3.

The analysis described here is far from complete. We list here a couple of issues that seem to us particularly worthy of attention:

- Only in one case (§4.1.1) did we find a locally bifurcating family of stable equilibrium solutions. In most cases, we find that orbits of our dynamical systems leave a small neighborhood of the origin and grow to large amplitude. In one case (§4.1.3) we know that this reflects the behavior of the fluid. In others it may not. It is possible that some of these behaviors could be captured by carrying the amplitude expansions to $k > 3$ in equation (13), but our present analyses exclude this.

- Generic normal forms for large, reversible systems exist but are not sufficient for our purpose: we need the specific normal forms involving, not generic parameters, but specific values of them belonging to the specific systems considered. These are yet to be calculated for the oscillatory onsets.

There have been other kinds of nonlinear analyses of the elliptic instability beyond those described above. Waleffe\textsuperscript{16} has considered an amplitude expansion directed toward understanding of a particular instability. The instability he studied is the cylindrical analog of the $2a \oplus 2b$ case described above, and is based on projection on the “active” normal modes of the linear problem. Gledzer\textsuperscript{6} has carried out a Galerkin truncation very much like that described here, but limited to the twelve basis functions that are at most quadratic polynomials
in the cartesian coordinates (thereby excluding the possibility of capturing the
weakly nonlinear behavior in most cases).

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icial computations were carried out using LSODE\textsuperscript{17}. 

REFERENCES

Table 1: Symmetry classes. As an example, a vector field in class 1 has an $x_1$-component that is odd in $x_1$ but even in $x_2$ and $x_3$, an $x_2$-component that is odd in $x_2$ but even in $x_1$ and $x_3$ and an $x_3$-component that is even in $x_1$ and $x_2$ and odd in $x_3$.

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FIGURE CAPTIONS

Figure 1: Six wedges of instability are shown. Axially symmetric ($\epsilon = 0$) flows are stable, but thin wedges of instability widen as $\epsilon$ increases.

Figure 2: (a) bifurcation diagram for equilibrium solutions of the Hamiltonian system determined by the Hamiltonian (6). (b)phase portraits for normal-forms reversible systems at an oscillatory onset projected onto the plane of the amplitudes $r_0$ and $r_1$; they depend on the sign of a parameter.

Figure 3: The perturbation of the equatorial circulation as determined from dynamical systems in class $1 \oplus 3c$ of dimensions eight, twenty-five and fifty-six (labeled ord1, ord2 and ord3 respectively). The top diagram compares the first two of these for about seventy-five time units, the bottom diagram compares the second two for 125 time units, all with the same initial data.
FIGURES

Fig.1, Lebovitz, Physics of Fluids:
Fig. 2, Lebovitz, Physics of Fluids:
Fig 3., Lebovitz, Physics of Fluids: